Math Strategies

Improving Mathematical Problem-Solving

Grades 4th–8th

RECOMMENDATION 1

Prepare problems and use them in whole-class instruction.

RECOMMENDATION 2

Assist students in monitoring and reflecting on the problem-solving process.

RECOMMENDATION 3

Teach students how to use visual representations.

RECOMMENDATION 4

Expose students to multiple problem-solving strategies.

RECOMMENDATION 5

Help students recognize and articulate math concepts and notation.

This document provides a summary of recommendations 1–5 from the WWC Practice Guide *Improving Mathematical Problem Solving in Grades 4 Through 8* (Woodward et al., 2018).

Content Area	Level of Evidence	Grades
Math	Minimal	4th 5th 6th 7th 8th

RECOMMENDATION 1

Prepare problems and use them in whole-class instruction.

Teachers should set aside time for problem-solving activities with the entire class instead of limiting problem-solving to individual homework assignments and include a variety of problems in these activities. Additionally, teachers should ensure that students understand the language, context, and math concepts of the problems included in lessons and homework.

Strategies

1. Include both routine and nonroutine problems in problem-solving activities.

Instructional strategies from the examples:

- Align use of routine or nonroutine problems with students' previous experience with problem-solving.
- Routine problems can be solved using familiar methods, replicating previously learned methods in a step-by-step fashion.

Nonroutine problems involve approaches that are not as predictable or well rehearsed; solution pathways aren't explicitly suggested by the task, task instructions, or in a worked-out example. Routine problems can be solved using approaches that students have already learned. Nonroutine problems, on the other hand, require using approaches that students are less familiar with or that are less obvious from the problem. When the goal of a lesson is to help students understand the meaning of an operation or mathematical idea, teachers should select routine problems. These do not necessarily have to be simple—they can be complex, multistep problems that involve problem-solving approaches students are already familiar with. When the goal of a lesson is to develop students' ability to think strategically, teachers should select nonroutine problems.

Examples of Routine Problems

Likely routine for a student who has studied and practiced multiplication with mixed numbers:

Carlos is following a cookie recipe that calls for $1\ 2/3$ cups of flour. He needs to make 4 batches of cookies. How much flour does he need?

Likely routine for a student who has studied and practiced solving linear equations with one variable:

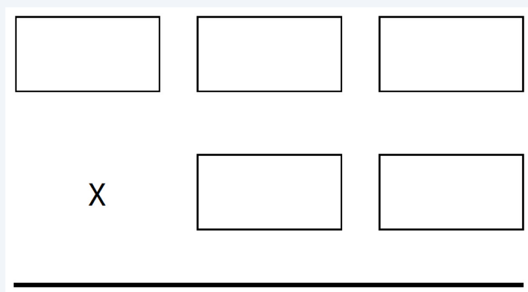
Solve for x: $20+8x=60$

Note. Adapted from Example 1 on page 12 of the practice guide (Woodward et al., 2018).

Examples of Nonroutine Problems

Likely nonroutine for students who are solidifying their understanding of multiplication:

The digits 1, 2, 3, 4, and 5—using each of these digits only once—are arranged in the blank boxes in the template below. Of all the possible arrangements of the digits 1 through 5, which one will produce the largest possible product? Why does this arrangement work?



Likely nonroutine for students in beginning algebra:

There are 20 people in a room. Everybody shakes the hand of everyone else.
How many handshakes occurred?

Note. Adapted from Example 2 on page 12 of the practice guide (Woodward et al., 2018).

2. Ensure that students will understand the problem by addressing issues students might encounter with the problem's context or language.

Instructional strategies from the examples:

- Explain contexts or vocabulary that may be unfamiliar to ensure students understand the language and context of problems—not to make problems less challenging, but to allow students to focus on the mathematics in the problem rather than on the need to learn new background knowledge or language.

The problems a teacher selects for a lesson may include unfamiliar vocabulary or contexts, making it challenging for students to focus on the math content. This is a particularly critical issue for English learners and students with disabilities. To ensure students' understanding without lessening the mathematical challenge, teachers can:

- Choose problems with language or contexts that are appropriate for the students' background.
- Clarify unfamiliar language or contexts in existing problems.
- Reword problems that contain unfamiliar words or phrases for students.

Examples of Clarifying Vocabulary and Context

Example Problem	Vocabulary	Context
In a factory, 54,650 parts were made. When they were tested, 4% were found to be defective. How many parts were working?	Students need to understand the term defective as being the opposite of working and the symbol % as percent to correctly solve the problem.	What is a factory? What does "parts" mean in this context?
At a used-car dealership, a car was priced at $7,000. The salesperson then offered a discount of $350. What percent discount, applied to the original price, gives the offered price?	Students need to know what offered and original price mean to understand the goal of the problem, and they need to know what discount and percent discount mean to understand what mathematical operators to use.	What is a used-car dealership?

Note. Taken from Example 3 on page 14 of the practice guide (Woodward et al., 2018).

3. Consider students' knowledge of math content when planning lessons.

Instructional strategies from the examples:

- Review relevant skills and knowledge needed to understand and solve a problem, especially if the mathematical content has not been discussed recently or if a nonroutine, challenging problem is presented.

Teachers should consider the concepts, skills, and vocabulary their students will need to solve problems included in lessons. For example, when finding the area of a circle, students may need to review the definitions of radius and pi as well as the concepts of perimeter and area. A brief review of the skills and vocabulary needed to understand and solve a problem may not only benefit struggling students but also help all students see how the knowledge they already have applies to more challenging problems.

Example of Reviewing Mathematical Language

Problem
Two vertices of a triangle are located at $(0, 4)$ and $(0, 10)$. The area of the triangle is 12 square units. What are all possible positions for the third vertex?
Mathematical Language to Review
• Vertices • Area square units • Triangle • Vetrex

Note. Taken from Example 5 on page 15 of the practice guide (Woodward et al., 2018).

Potential Roadblocks

Teachers are having trouble finding problems for the problem-solving activities.

Suggested Approach. Teachers can reference supplementary materials (for example, books on problem-solving), ask colleagues for additional problem-solving activities, or search the internet for examples. Useful resources on the internet include "Problems of the Week" from the Math Forum (https://www.nctm.org/pows/), "Illuminations" from the National Council of Teachers of Mathematics (https://illuminations.nctm.org/), and practice problems from standardized tests such as the PISA (http://www.oecd.org/pisa/; PISA 2012 Mathematics Items), SAT (https://collegereadiness.collegeboard.org/sat/practice), or TIMSS (TIMSS—Released Assessment Questions).

Teachers have no time to add problem-solving activities to their math instruction.

Suggested Approach. To make time during lessons, teachers can replace some of the problems students are required to solve during seatwork with fully solved problems that students can review and use as problem-solving models.

Teachers are not sure which words to teach when teaching problem-solving.

Suggested Approach. Math coaches and specialists can provide lists of words and phrases essential for teaching a given unit. Teachers can also work with colleagues to identify words students need to understand and solve problems. They can also look for important terms in class textbooks or state math standards.

Content Area	Level of Evidence	Grades
Math	Strong	4th 5th 6th 7th 8th

RECOMMENDATION 2

Assist students in monitoring and reflecting on the problem-solving process.

Considering what students are doing and why they are doing it during the problem-solving process helps them learn math more effectively. It helps students assess the path they take while solving a problem and helps them connect knowledge they already have to newer concepts. Teachers can help students in this process through asking guiding questions, modeling self-monitoring and reflection, and building on students' own reflections to help them improve their problem-solving.

Strategies

1. Provide students with a list of prompts to help them monitor and reflect during the problem-solving process.

Instructional strategies from the examples:

- Provide questions that students should ask and answer as they solve problems.
- Provide task lists that help students complete steps in the problem-solving process.
- Encourage students to explain and justify their response to each prompt, either orally or in writing.

Teachers can provide students with two types of prompts: (1) questions students should ask themselves, and (2) task lists that students can follow each time they work through solving a problem. When first introducing prompts, teachers may need to help students understand how to use them. Teachers can help students by talking through the questions or task lists in whole-class or small-group discussions. If students obtain an incorrect solution, teachers can provide the correct solution and ask students to explain why that answer is correct (and, conversely, why their original solution is wrong). The more competent students become at problem-solving, the less support teachers will need to give. Teachers should be sure not to overburden students with long lists of prompts. Doing so may lead students to solve problems more slowly or abandon the prompts altogether.

Two Types of Prompts

Example Questions	Example Task List
• What is this problem asking? • What do I know about the problem so far? What information is given to me? How can this help me? • Which information in this prompt is relevant to solving the problem? • What are some different ways I could approach solving this problem? • Why did this approach work? • Why didn't it work?	1. Figure out what the question is asking for and what information is given. 2. Identify the type of problem. 3. Recall solutions to previous problems that may be useful in the current problem. 4. Make a visual to help represent or solve the problem. 5. Solve the problem. 6. Check your solution.

Note. Adapted from Examples 6 and 7 on page 19 of the practice guide (Woodward et al., 2018).

2. Model how to monitor and reflect on the problem-solving process.

Instructional strategies from the examples:

- As you work through a problem, say not only the response to each prompt but also the reason(s) why you took each.
- Alternatively, as you work a step in solving the problem, ask students to explain why this works.

Using prompts, teachers can show how to monitor and reflect during the problem-solving process. Teachers can give students an appropriate response to each prompt and either explain the reasoning behind that response or ask the students to explain why that response makes sense. Teachers should ensure that each step of the problem-solving process is represented by a prompt.

Example of Modeling How to Monitor and Reflect

Problem
Last year was unusually dry in Colorado. Denver usually gets 60 inches of snow per year. Vail, which is up in the mountains, usually gets 350 inches of snow. Both places had 10 inches of snow less than the year before. Kara and Ramon live in Colorado and heard the weather report. Kara thinks the decline for Denver and Vail is the same. Ramon thinks that when you compare the two cities, the decline is different. Explain how both people are correct.

Solution
TEACHER: First, I ask myself, "What is this story about, and what do I need to find out?" I see that the problem has given me the usual amount of snowfall and the change in snowfall for each place, and that it talks about a decline in both cities. I know what decline means: "a change that makes something less." Now I wonder how the decline in snowfall for Denver and Vail can be the same for Kara and different for Ramon. I know that a decline of 10 inches in both cities is the same, so I guess that's what makes Kara correct. How is Ramon thinking about the problem? I ask myself, "Have I ever seen a problem like this before?" As I think back to the assignments we had last week, I remember seeing a problem that asked us to calculate the discount on a $20 item that was on sale for $15. I remember we had to determine the percent change. This could be a similar kind of problem. This might be the way Ramon is thinking about the problem. I ask myself, "What steps should I take to solve this problem?" It looks like I need to divide the change amount by the original amount to find the percent change in snowfall for both Denver and Vail. Denver: $10 \div 60 = 0.166$ or 16.67% or 17% when we round it to the nearest whole number. Vail: $10 \div 350 = 0.029$ or 2.9% or 3% when we round it to the nearest whole number. So the percent decrease in snow for Denver was much greater (17%) than for Vail (3%). Now I see what Ramon is saying! It's different because the percent decrease for Vail is much smaller than it is for Denver. Finally, I ask myself, "Does this answer make sense when I reread the problem?" Kara's answer makes sense because both cities did have a decline of 10 inches of snow. Ramon is also right because the percent decrease for Vail is much smaller than it is for Denver. Now, both of their answers make sense to me.

Note. Taken from Example 8 on page 20 of the practice guide (Woodward et al., 2018).

3. Use student thinking about a problem to develop students' ability to monitor and reflect.

Instructional strategies from the examples:

- Help students verbalize other ways to think about the problem.
- Include guided questioning to help students clarify and refine their thinking or establish a method for monitoring and reflecting that makes sense to them.

Teachers can help students establish methods for monitoring and reflecting that make sense to students. Teachers can establish a dialogue with students that includes guiding questions to clarify and refine their thinking. This activity is helpful for students who dislike, or have trouble understanding, teacher-provided prompts.

Example of Using Student Ideas to Clarify and Refine the Monitoring and Reflecting Process

Problem
Find a set of five different numbers whose average is 15.
Solution
TEACHER: Jennie, what did you try? **STUDENT**: I'm guessing and checking. I tried 6, 12, 16, 20, 25, and they didn't work. The average is like 17.8 or something decimal like that. **TEACHER**: That's pretty close to 15, though. Why'd you try those numbers? **STUDENT**: What do you mean? **TEACHER**: I mean, where was the target, 15, in your planning? It seems like it was in your thinking somewhere. If I were choosing five numbers, I might go with 16, 17, 20, 25, 28. **STUDENT**: But they wouldn't work—you can tell right away. **TEACHER**: How? **STUDENT**: Because they are all bigger than 15. **TEACHER**: So? **STUDENT**: Well, then the average is going to be bigger than 15. **TEACHER**: Okay. That's what I meant when I asked, "Where was 15 in your planning?" You knew they couldn't all be bigger than 15. Or they couldn't all be smaller either? **STUDENT**: Right. **TEACHER**: Okay, so keep the target, 15, in your planning. How do you think five numbers whose average is 15 relate to the number 15? **STUDENT**: Well, some have to be bigger and some smaller. I guess that's why I tried the five numbers I did. **TEACHER**: That's what I guess, too. So the next step is to think about how much bigger some have to be and how much smaller the others have to be. Okay? **STUDENT**: Yeah. **TEACHER**: So use that thinking to come up with five numbers that work.

Note. Taken from Example 9 on page 21 of the practice guide (Woodward et al., 2018).

Potential Roadblocks

Students don’t want to monitor and reflect; they just want to solve the problem.

Suggested Approach. Explain to students that getting into the habit of monitoring and reflecting every time they solve a problem will improve their problem-solving abilities, and that monitoring and reflecting are still an integral part of the process for experienced problem-solvers.

Teachers are unclear on how to think aloud while solving a non-routine problem.

Suggested Approach. Teachers can prepare ahead of time by creating outlines of responses to prompts and by anticipating how students may think about prompts. They can seek help from colleagues or math coaches if they get stuck.

Students take too much time to monitor and reflect on the problem-solving process.

Suggested Approach. While students may solve problems slowly when they begin learning how to monitor and reflect, they will become more efficient with practice.

When students reflect on the problems they have already solved, they resort to using methods from problems rather than adapting their efforts to the new problem before them.

Suggested Approach. Ask students to explain why their solution worked in the previous problem, and why it may or may not work for the current problem.

Content Area	Level of Evidence	Grades
Math	Strong	4th 5th 6th 7th 8th

RECOMMENDATION 3

Teach students how to use visual representations.

Translating quantitative information into an algebraic or arithmetic form is a critical component of the problem-solving process. Students who learn how to represent this information visually before translating it into an algebraic or arithmetic form tend to be more effective at problem-solving. When teaching students to use visual representations (e.g., graphs, diagrams, number lines, tables), teachers should choose visuals that are appropriate for the problem at hand and their students, then use them consistently for similar problems so as not to overwhelm students with too many differing examples.

Sample Visual Representations

Strip diagrams use rectangles to represent quantities presented in the problem.

Percent bars are strip diagrams in which each rectangle represents a part of 100 in the problem.

Schematic diagrams demonstrate the relative sizes and relationships between quantities in the problem.

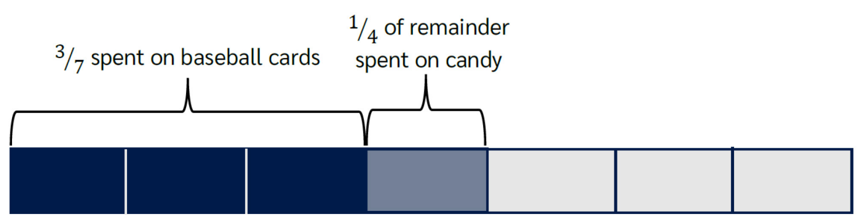
Below are two examples of how visual representations might be used to solve problems. For additional examples, see page 24 of the practice guide (Woodward et al., 2018).

Example 1

Problem

Ben spent 3/7 of his allowance on baseball cards and then 1/4 of what remained on candy. After this, he had $50 left. How much did he start with?

Sample Strip Diagram



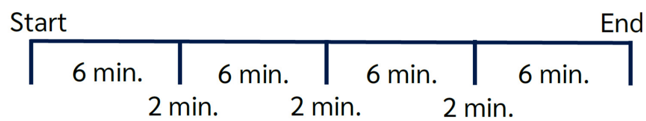
This diagram depicts the original amount of money that Ben had, divided into 7 equal parts.

Problem
It can be seen that 3 of the 7 parts have not been spent. From this diagram, there are multiple approaches to creating an equation.

Note. Adapted from Example 10 on page 24 of the practice guide (Woodward et al., 2018).

Example 2

Problem
Jackie recently ran 4 laps. She usually runs 1 lap in 6 minutes. Jackie took a 2-minute break after every lap. How much time did it take her to complete the 4-lap run?
Sample Strip Diagram



This diagram illustrates Jackie running 4 laps, with each lap taking 6 minutes. As she runs, Jackie needs to take a 2-minute break between each lap. From this diagram, an equation, such as $(6\times4)+(2\times3)=x$, can be created to find the total number of minutes Jackie takes to run 4 laps.

Note. Adapted from Example 10 on page 25 of the practice guide (Woodward et al., 2018).

Strategies

1. Select visual representations that are appropriate for students and the problems they are solving.

Instructional strategies from the examples:

- Use schematic diagrams with ratio and proportion problems.
- Use percent bars for percent problems.
- Use strip diagrams for comparison and fraction problems.

Rather than using all visual representations recommended for a particular type of problem, teachers should select the visual representations they think will work best for their students. Teachers should use selected representations consistently for similar problems so as not to overwhelm students with too many examples and give students time to learn how to successfully use the selected representations. If students still struggle with a representation after a reasonable amount of time, teachers should consider using a different type of representation.

2. Use think-alouds and discussions to teach students how to represent problems visually.

Instructional strategies from the examples:

- Demonstrate how to represent the problem using the representation, using think-alouds to describe the decisions you make to connect the problem to the representation.
- Explain how to identify the problem type based on mathematical ideas in the problem and why a certain representation is most appropriate.
- Teach students to identify what information is relevant or critical to solving the problem.
- Encourage students to discuss similarities or differences among visuals they have used.

Teachers should demonstrate the thought process behind connecting a problem to a visual representation by thinking aloud when explaining a new representation. Thinking aloud goes beyond teachers telling students what they are doing; it involves teachers explaining why they are taking the particular steps. Teachers should explain how they identified the type of math problem and why they think the selected representation is appropriate for that problem. They should demonstrate how to identify the information in a problem that is relevant to solving it.

Example of Using a Think-Aloud

Problem
Monica and Bianca went to a flower shop to buy some roses. Bianca bought a bouquet with 5 pink roses. Monica bought a bouquet with two dozen roses, some red and some yellow. She has 3 red roses in her bouquet for every 5 yellow roses. How many red roses are in Monica's bouquet?

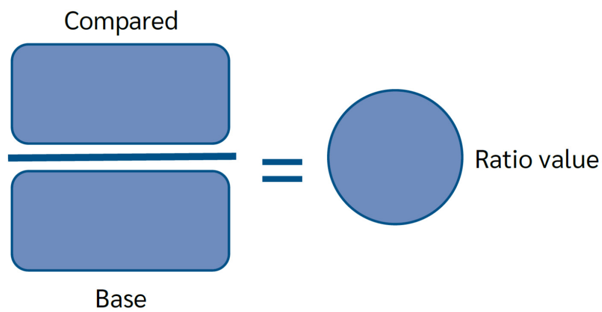
Sample Strip Diagram

TEACHER: I know this is a ratio problem because two quantities are being compared: the number of red roses and the number of yellow roses. I also know the ratio of the two quantities. There are 3 red roses for every 5 yellow roses. This tells me I can find more of each kind of rose by multiplying.

I reread the problem and determine that I need to solve the question posed in the last sentence: "How many red roses are in Monica's bouquet?" Because the question is about Monica, perhaps I don't need the information about Bianca. The third sentence says there are two dozen red and yellow roses. A dozen is 12. I know that makes 24 red and yellow roses, but I still don't know how many red roses there are. I know there are 3 red roses for every 5 yellow roses. I think I need to figure out how many red roses there are in the 24 red and yellow roses.

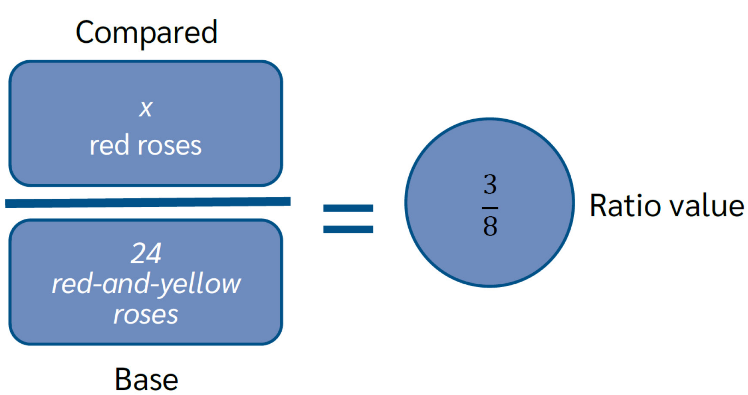
Sample Strip Diagram

Let me reread the problem...That's correct. I need to find out how many red roses are in the bouquet of 24 red and yellow roses. The next part of the problem talks about the ratio of red roses to red and yellow roses. I can draw a diagram that helps me understand the problem. I've done this before with ratio problems. These kinds of diagrams show the relationship between the two quantities in the ratio.



I write the quantities and units from the problem and an x for what must be solved in the diagram. First, I am going to write the ratio of red roses to yellow roses here in the circle. This is a part-to-whole comparison—but how can I find the whole in the part-to-whole ratio when we only know the part-to-part ratio (the number of red roses to the number of yellow roses)?

I have to figure out what the ratio is of red roses to red and yellow roses when the problem only tells me about the ratio of red roses to yellow roses, which is 3:5. So if there are 3 red roses for every 5 yellow roses, then the total number of units for red and yellow roses is 8. For every 3 units of red roses, there are 8 units of red and yellow roses, which gives me the ratio 3:8. I will write that in the diagram as the ratio value of red roses to red and yellow roses. There are two dozen red and yellow roses, and that equals 24 red and yellow roses, which is the base quantity. I need to find out how many red roses (x) there are in 24 red and yellow roses.



Sample Strip Diagram
I can now translate the information in this diagram to an equation like this: $$\frac{x \text{ red roses}}{24 \text{ red-and-yellow roses}} = \frac{3}{8}$$ Then, I need to solve for x. $$\frac{x}{24} = \frac{3}{8}$$ $$24\left(\frac{x}{24}\right) = 24\left(\frac{3}{8}\right)$$ $$x = \frac{72}{8}$$ $$x = 9$$

Note. Taken from Example 11 on pages 27–28 of the practice guide (Woodward et al., 2018).

3. Show students how to convert the visually represented information into mathematical notation.

Instructional strategies from the examples:

- Show students how each quantity and relationship in the visual representation corresponds to those in the equation.

Teachers should show students how to translate quantities and relationships in visual representations into math equations. Sometimes, all this translation requires is removing boxes, arrows, and other visual elements from the representation. With more complicated examples, teachers may need to provide more explicit illustrations of the connection between the representations and mathematical notation.

Potential Roadblocks

Students do not capture the relevant details in the problem or include unnecessary details when representing a problem visually.

Suggested Approach. If students are missing relevant details in their visual representations, teachers can ask guiding questions to build on students' thinking and refine their representations. Once the representations are refined, teachers can ask students why their initial representations did not work. If guiding questions do not work, teachers can demonstrate how to alter students' representations to represent the problems appropriately and eliminate unnecessary detail. Teachers should point out elements of the representations that were done correctly so students are encouraged to continue trying.

The class text does not use visual representations.

Suggested Approach. Teachers can incorporate visual representations into lessons using media such as whiteboards, overhead projectors, or interactive whiteboards. Teachers can tap colleagues or math coaches for useful visual representations or develop their own. The internet and professional development materials may have useful examples.

Content Area	Level of Evidence	Grades
Math	Moderate	4th 5th 6th 7th 8th

RECOMMENDATION 4

Expose students to multiple problem-solving strategies.

Students who learn how to use multiple problem-solving strategies can approach problems with greater ease and flexibility in finding solutions. Teachers should demonstrate that problems can be solved in multiple ways and that approaches to problem-solving should be selected for their ease and efficiency. Teachers can present students with a number of problem-solving strategies and then give them the opportunity to compare, contrast, and carry out the strategies.

Strategies

1. Provide instruction in multiple strategies.

Instructional strategies from the examples:

- Demonstrate multiple ways to solve the same problem.
- At times, use an unsuccessful strategy and demonstrate changing to an alternate strategy.

Teachers can present strategies for general use as well as those for specific problems. When demonstrating different strategies, teachers should occasionally attempt unsuccessful strategies and then change to successful ones. Doing so will show students that some problems may not be easy to solve the first time and they may need to try more than one strategy before successfully solving a problem. Teachers should also demonstrate more than one successful approach to the same problem.

Example of Two Ways to Solve the Same Problem

Problem
Ramona's furniture store has a choice of 3-legged stools and 4-legged stools. There are five more 3-legged stools than 4-legged stools. When you count the legs of the stools, there are exactly 29 legs. How many 3-legged and 4-legged stools are there in the store?

Solution 1: Guess and Check		
4x4 legs = 16 legs	9x3 legs = 27 legs	Total: 43 legs
3x4 legs = 12 legs	8 x 3 legs = 24 legs	Total = 36 legs
2x4 legs = 8 legs	7x3 legs = 21 legs	Total: 29 legs

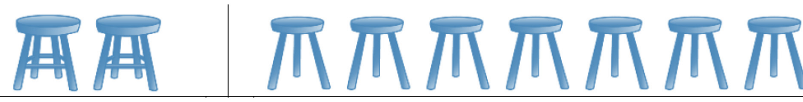
TEACHER: This works; the total equals 29, and with two 4-legged stools and seven 3-legged stools, there are five more 3-legged stools than 4-legged stools.

Solution 2

TEACHER: Let's see if we can solve this problem logically. The problem says that there are five more 3-legged stools than 4-legged stools. It also says that there are 29 legs altogether. If there are five more 3-legged stools, there has to be at least one 4-legged stool in the first place. Let's see what that looks like.

Stools			
Total legs	$4 \times 1 = 4$	$+$	$3 \times 6 = 18$
		$4 + 18 = 22$	

TEACHER: We can add a stool to each group, and there will still be a difference of five stools.



Stools			
Total legs	$4 \times 2 = 8$	$+$	$3 \times 7 = 21$
		$8 + 21 = 29$	

TEACHER: I think this works. We have a total of 29 legs, and there are still five more 3-legged stools than 4-legged stools. We solved this by thinking about it logically. We knew there was at least one 4-legged stool and there were six 3-legged stools. Then we added to both sides so we always had a difference of five stools.

Note. Taken from Example 14 on page 34 of the practice guide (Woodward et al., 2018).

2. Provide opportunities for students to compare multiple strategies in worked examples.

Instructional strategies from the examples:

- Ask students to compare the similarities and differences among multiple strategies.
- Provide opportunities for students to work with a partner to discuss strategies in worked examples.
- Use worked examples alongside opportunities for students to solve problems on their own.

Teachers should provide side-by-side examples of different problem-solving strategies and give students the opportunity to work together to compare the strategies. Teachers can prompt students to compare and contrast the strategies, to justify the approach they would choose, and to consider why two different approaches can lead to the same answer. Teachers can provide worked examples alongside problems that students are required to solve. Students can practice describing their solution paths both verbally and in writing.

Example of Comparing Strategies

Sanjin's Solution		Emily's Solution	
$7(x-3)=4(x-3)-3$		$7(x-3)=4(x-3)-3$	
$7x-21=4x-12-3$	Distribute	$3(x-3)=-3$	Subtract on both
$7x-21=4x-15$	Combine	$x-3=-1$	Divide on both
$3x-21=-15$	Subtract on both	$x=2$	Add on both
$3x=6$	Add on both		
$x=2$	Divide on both		

TEACHER: Sanjin and Emily used different approaches but got the same answer. Why is this? Which of their approaches would you choose?

Note. Adapted from Example 15 on page 35 of the practice guide (Woodward et al., 2018).

3. Ask students to generate and share multiple strategies for solving a problem.

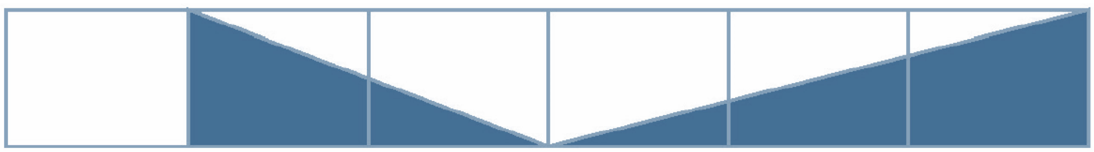
Instructional strategies from the examples:

- Rather than randomly calling on students to share their strategies, select students purposefully based on the strategies they have used to solve the problem.

Teachers should encourage students to generate multiple problem-solving strategies and give them opportunities to share their strategies with the class. Rather than calling on students randomly, teachers should call on students who generated strategies different from the one presented. Teachers should encourage students to not only present their approach but also to explain why they chose that approach. Examples 16 and 17 on pages 36–37 of the practice guide provide excellent models for teachers.

Example of Two Students Sharing Their Strategies for Solving a Fractions Problem

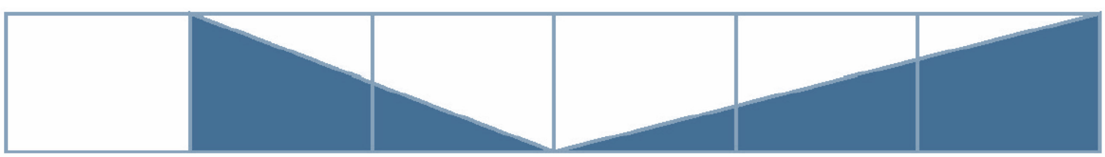
Problem
What fraction of the whole rectangle is blue?



Solution

STUDENT 1: If I think of it as what's to the left of the middle plus what's to the right of the middle, then I see that on the left, the blue part is $1/3$ of the area, so that is $1/3$ of $1/2$ of the entire rectangle. On the right, the blue part is $1/2$ of the area, so it is $1/2$ of $1/2$ of the entire rectangle. This information tells me that the blue part is

$(1/3 \times 1/2) + (1/2 \times 1/2) = 1/6 + 1/4 = 2/12 + 3/12 = 5/12$ of the entire rectangle.



STUDENT 2: I see that the original blue part and the part I've colored dark blue have the same area. So the original blue part is $1/2$ of the dark-blue-and-blue part, or $1/2$ of $5/6$ of the entire rectangle.

This tells me that the original blue part is $1/2 \times 5/6 = 5/12$ of the entire rectangle.

Note. Taken from example 17 on page 37 of the practice guide (Woodward et al., 2018).

Potential Roadblocks

Teachers don’t have enough time in their math class for students to present and discuss multiple strategies.

Suggested Approach. Teachers can ask students to write out their strategies on personal whiteboards or chart paper so they do not have to spend time rewriting them on the class whiteboard. Another approach is for teachers to document the strategies students come up with during independent or small-group work and summarize them for the class.

Not all students are willing to share their strategies.

Suggested Approach. Teachers should encourage students to share their strategies even if they are incorrect. They should explain to students that there may be a variety of approaches to solving most problems and that the solutions they share may not have been considered by the other students. Teachers can point out that sharing will help students learn effective problem-solving strategies from one another.

Some students struggle to learn multiple strategies.

Suggested Approach. If students lack or cannot retrieve necessary knowledge, they may struggle with using multiple strategies. Teachers may need to ask students to write down the facts of a problem before trying to solve it. They may need to modify a problem to make it easier to focus on problem-solving (rather than the arithmetic, for example). Teachers can also write down problem solutions side by side so that students can more easily compare the solutions.

Some of the strategies students share are not clear or do not make sense to the class.

Suggested Approach. Teachers can walk around the classroom and ask students to explain their approaches individually. Doing so will better prepare teachers to clarify students’ thinking when they share their approaches with the class, either by asking guiding questions or by rewording the students’ approaches. Teachers can also ask other students to restate what a student said.

Content Area	Level of Evidence	Grades
Math	Moderate	4th 5th 6th 7th 8th

RECOMMENDATION 5

Help students recognize and articulate math concepts and notation.

Students who have a solid understanding of math concepts and notation are better able to identify the math content of a problem, adapt their current knowledge to new problems, and consider multiple approaches to solving problems. Teachers can convey math concepts and notation by using problem-solving activities, by asking students to use mathematically valid explanations to explain how a worked problem was solved, and by introducing students to algebraic notation in a systematic manner.

Strategies

1. Describe relevant math concepts and notation, and relate them to the problem-solving activity.

Instructional strategies from the examples:

- Watch and listen for opportunities to call attention to mathematical concepts and notation students use as they solve problems.
- Draw attention to mathematical ideas and concepts by directly instructing students in them before engaging the students in problem-solving.

Teachers can help students learn to connect their existing math intuition to formal concepts and notation. Teachers can do this by watching and listening while students solve problems and pointing out formal concepts and notations as they are used. If students use informal language or notation for a concept, teachers can translate it into formal mathematical language and explain that there are multiple ways to communicate the same idea, moving students toward more formal mathematical approaches. Occasionally, teachers may need to explicitly instruct students in formal math concepts and notation before proceeding to problem-solving.

Example of Students' Intuitive Understanding of Formal Math Concepts

Problem
Is the sum of two consecutive numbers always odd?

Sample Strip Diagram

STUDENT: Yes.

TEACHER: How do you know?

STUDENT: Well, suppose you take a number, like 5. The next number is 6.

For 5, I can write five lines, like this:

| | | | |

For 6, I can write five lines and one more line next to it, like this:

| | | | | |

Then, I can count all of them, and I get 11 lines. See? It's an odd number.

TEACHER: When you say, "It's an odd number," you mean the sum of the two consecutive numbers is odd. So can you do that with any whole number, like n? What would the next number be?

STUDENT: It would be $n + 1$.

TEACHER: So can you line them up like you did for 5 and 6?

STUDENT: You mean, like this?

n

$n + 1$

TEACHER: Right. So what does that tell you about the sum of n and $n + 1$?

STUDENT: It's 2 ns and 1, so it's odd.

TEACHER: Very good. The sum, which is $n + n + 1 = 2n + 1$, is always going to be odd.

Note. Taken from example 18 on page 41 of the practice guide (Woodward et al., 2018).

2. Ask students to explain each step used to solve a problem in a worked example.

Instructional strategies from the examples:

- Provide students with opportunities to explain the process used to solve a problem in a worked example and to explain why the steps worked.
- Use small-group activities to encourage students to discuss the process used in a worked example and the reasoning for each step.
- Use probing questions to help students articulate mathematically valid explanations.

Teachers should give students an opportunity to explain what problem-solving process they used in a worked example as well as why that process worked. Students can discuss the problem-solving process in small groups or restate one another's explanations in pairs. If students struggle to come up with mathematically valid explanations, teachers should ask questions to guide students to more valid explanations. Teachers also can provide examples of valid explanations or reword students' explanations.

Example of Working through a Sample Strip Diagram

Problem

Are 2/3 and 8/12 equivalent fractions?

Sample Strip Diagram

STUDENT: To find an equivalent fraction, whatever we do to the top of 2/3 we must also do to the bottom.

TEACHER: What do you mean?

STUDENT: It just works when you multiply it.

TEACHER: What happens when you multiply in this step?

STUDENT: The fraction…stays the same.

TEACHER: That's right. When you multiply a numerator and denominator by the same number, you get an equivalent fraction. Why is that?

STUDENT: Before there were 3 parts, but we made 4 times as many parts, so now there are 12 parts.

TEACHER: Right, you had 2 parts of a whole of 3. Multiplying both by 4 gives you 8 parts of a whole of 12. That is the same part-whole relationship—the same fraction, as you said. Here's another way to look at it: When you multiply the fraction by 4/4, you are multiplying it by a fraction equivalent to 1; this is the identity property of multiplication, and it means when you multiply anything by 1, the number stays the same.

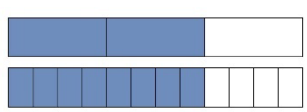
A Correct Description, But Still Not a Complete Explanation

STUDENT: Whatever we multiply the top of 2/3 by we must also multiply the bottom by.

A Mathematically Valid Explanation

STUDENT: You can get an equivalent fraction by multiplying the numerator and denominator of 2/3 by the same number. If we multiply the numerator and denominator by 4, we get 8/12.

If I divide each of the third pieces in the first fraction strip into 4 equal parts, then that makes 4 times as many parts that are shaded and 4 times as many parts in all. The 2 shaded parts become $2 \times 4 = 8$ smaller parts, and the 3 total parts become $3 \times 4 = 12$ total smaller parts. So the shaded amount is 2/3 of the strip, but it is also 8/12 of the strip:



Note. Adapted from Example 19 on page 42 of the practice guide (Woodward et al., 2018).

3. Help students make sense of algebraic notation.

Instructional strategies from the examples:

- Introduce symbolic notation early and at a moderate pace, allowing students enough time to become familiar and comfortable with it.
- Ask students to explain each component of an algebraic equation by having them link the equation back to the problem they are solving.

To allow students time to become comfortable with the symbolic notation used in algebra, teachers should introduce symbolic notation early and at a rate that is not too slow or too fast for learners. One way of accomplishing this is to provide students with arithmetic problems and then support them in translating the problems into algebraic notation. This approach will help students connect their existing arithmetic knowledge with new algebraic knowledge.

Example of Linking Components of an Equation to a Problem

Problem
Joseph earned money for selling 7 CDs and his old headphones. He sold the headphones for \$10. He made \$40.31. How much did he sell each CD for?
Solution
The teacher writes this equation: $10+7x=40.31$ **TEACHER:** If x represents the number of dollars he sold the CD for, what does the $7x$ represent in the problem?   What does the 10 represent?   What does the 40.31 represent?   What does the $10+7x$ represent?

Note. Taken from Example 21 on page 43 of the practice guide (Woodward et al., 2012).

Potential Roadblocks

Students' explanations are too short and lack clarity and detail. It is difficult for teachers to identify which mathematical concepts they are using.

Suggested Approach. Teachers can determine what concepts students are likely to use by solving problems before using them in lessons. Teachers can ask students questions about how a problem was solved and how they thought about the problem. They can also ask students to make a sheet of mathematical rules and use those rules when explaining how they approached a specific problem. The sheet should be brief and include only a few key rules.

Students may be confused by mathematical notations used in algebraic equations.

Suggested Approach. Teachers should use, and encourage students to use, arbitrary variables in order to help students understand that variables play an abstract role in algebraic equations. For example, use x or y to represent an unknown quantity. Students may confuse variables if they seem related to items in a problem (for example, if a represents apples).

Reference

Woodward, J., Beckman, S., Driscoll, M., Franke, M., Herzig, P., Jitendra, A., Koedinger, K. R., & Ogbuehi, P. (2018). *Improving mathematical problem solving in grades 4 through 8* (NCEE 2012-4055). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/16