Math Strategies

Developing Effective Fractions Instruction

Grades K–8th

RECOMMENDATION 1

Build on students' informal understanding of sharing and proportionality to develop initial fraction concepts.

RECOMMENDATION 2

Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.

RECOMMENDATION 3

Help students understand why procedures for computations with fractions make sense.

RECOMMENDATION 4

Develop students' conceptual understanding of strategies for solving ratio, rate, and proportion problems before exposing them to cross-multiplication as a procedure to use to solve such problems.

This document provides a summary of recommendations 1–4 from the WWC Practice *Guide Developing Effective Fractions Instruction for Kindergarten through 8th Grade* (Siegler et al., 2010).

Content Area	Level of Evidence	Grades
Math	Moderate	K 1st 2nd 3rd 4th 5th 6th 7th 8th

RECOMMENDATION 1

Build on students' informal understanding of sharing and proportionality to develop initial fraction concepts.

Students develop a basic understanding of fractions before starting kindergarten through activities such as equally sharing a set of objects with a group of people and proportional reasoning. Teachers can build on this knowledge as they introduce students to the more formal concept of fractions (either as division or ratios), building on what students already understand about sharing equally. Although fraction concepts are usually not introduced until first or second grade, sharing activities can begin as early as PreK, setting the stage for a deeper understanding of fractions.

Equal sharing. By age 4, students can distribute equal numbers of equal-size objects among a small number of recipients, and the ability to equally share improves with age. Sharing a set of discrete objects (e.g., 12 grapes shared among three students) tends to be easier for young students than sharing a single object (e.g., a candy bar), but by age 5 or 6, students are reasonably skilled at both.

Proportional relations. By age 6, students can match equivalent proportions represented by different geometric figures and by everyday objects of different shapes. One-half is an important landmark in comparing proportions; students more often succeed with comparisons in which one proportion is more than half and the other is less than half than on comparisons in which both proportions are more than half, or both are less than half (e.g., comparing $1/3$ to $3/5$ is easier than comparing $2/3$ to $4/5$). In addition, students can complete analogies based on proportional relations—for example, a half-circle is to a half-rectangle as a quarter-circle is to a quarter-rectangle.

Strategies

1. Use equal-sharing activities to introduce the concept of fractions. Use sharing activities that involve dividing sets of objects as well as single whole objects.

Instructional strategies from the examples:

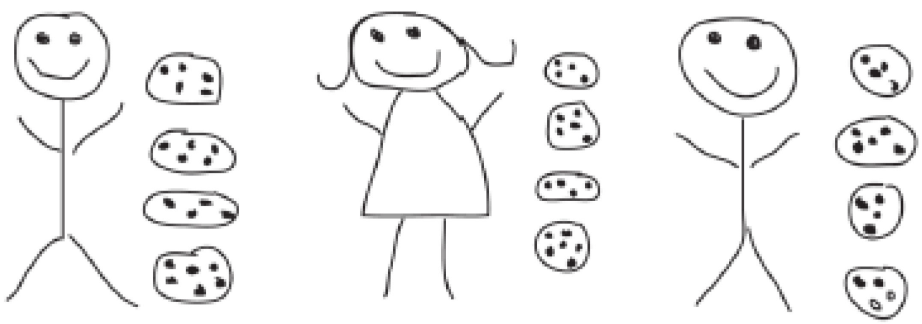
- Begin equal-sharing activities with a set of objects that can be evenly distributed between two people, then move to larger numbers of people.
- Once students have gained facility with equally sharing a group of objects, move to partitioning a single object into fractional parts for equal sharing.
- As teachers increase the number of people to share with, teachers should consider using multiples of two to allow students to use repeated halving to help solve problems. Later, teachers should move to activities that can't be solved with repeated halving.

Teachers should provide students with sharing activities that progressively build on their existing strategies for dividing, beginning with those involving equal sharing of a set of objects (first between two people, then increasing the number of people), then progressing to activities involving partitioning a set of objects, then a single object, into different fractional parts. Teachers should encourage students to use manipulatives (e.g., counters, beans) or drawings when doing these activities. As students work through activities and different representations, teachers can introduce formal fraction names and have students label fractions on their drawings. To most effectively build students' understanding, engage them in a variety of naming and labeling activities.

Sharing a set of objects. In early equal-sharing activities, teachers should have students solve problems that involve two people (then increase the number of people) and a set of objects that can be divided equally with none left over. To further develop students' understanding, teachers should present situations by describing the number of items to be shared and the number of people who will be sharing, then have students work to determine the number of items each person should receive. As students gain success, teachers can pose a similar problem with the same number of items but increase the number of people. Note that problems of this type should always focus on equal sharing among the recipients.

Example of Sharing a Set of Objects Evenly Among Recipients

Problem
Three students want to share 12 cookies so that each student receives the same number of cookies. How many cookies should each student get?
Example of Student Solution Strategy
Students can solve this problem by drawing three figures to represent the students and then drawing cookies by each figure, giving one cookie to the first student, one to the second, and one to the third, continuing until they have distributed 12 cookies to the three students, and then counting the number of cookies distributed to each student. Other students may solve the problem by simply dealing the cookies into three piles as if they were dealing cards.



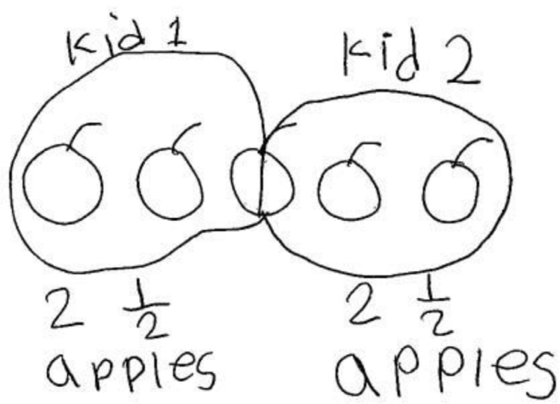
Note. Taken from Figure 1 on page 14 of the practice guide (Siegler et al., 2010).

Partitioning a single object. After students gain facility with equal-sharing activities, teachers should present problems that involve dividing one or more objects into equal parts. Thus, the focus of the problems shifts from how many objects each person has, to how much of an object each person should get. For initial problems, teachers should present one object to be shared between people, then progress to multiple objects being divided into smaller parts to be shared equally among people.

Problems involving people equally dividing one object result in unit fractions (e.g., $1/2$, $1/3$, $1/5$), while the more complex problems of multiple objects and multiple people often result in non-unit fractions (e.g., $1/2$).

Teachers should also consider sharing among groups that are multiples of two (e.g., $2, 4, 8, 16 \ldots$). This allows students to partition using repeated halving strategies. Later, teachers should introduce problems that cannot be solved with this strategy to help students develop other strategies. For example, teachers may provide students with toothpicks or wooden sticks to lay across objects to represent cutting or partitioning.

Example of Partitioning Both Multiple and Single Objects

Problem
Two students want to share five apples that are the same size so that both have the same amount to eat. Draw a picture to show what each student should receive.
Example of Student Solution Strategy
Students might solve this problem by drawing five circles to represent the five apples and two figures to represent the two students. Students might then draw lines connecting each student to two apples. Finally, they might draw a line partitioning the final apple into two approximately equal parts and draw a line from each part to the two students. Alternatively, as in the picture below, students might draw a large circle representing each student, two apples within each circle, and a fifth apple straddling the circles representing the two students. In yet another possibility, students might divide each apple into two parts and then connect five half apples to the representation of each figure. 

Note. Taken from Figure 2 on page 15 of the practice guide (Siegler et al., 2010).

2. Extend equal-sharing activities to develop students' understanding of ordering and equivalence of fractions.

Instructional strategies from the examples:

- As teachers gradually increase the number of people to share with, guide students to understand the inverse relationship between the number of people and the amount each person receives.
- Provide activities for students that also involve partitioning the number of sharers (e.g., move from one group sharing to the group now being divided equally at different tables and sharing).

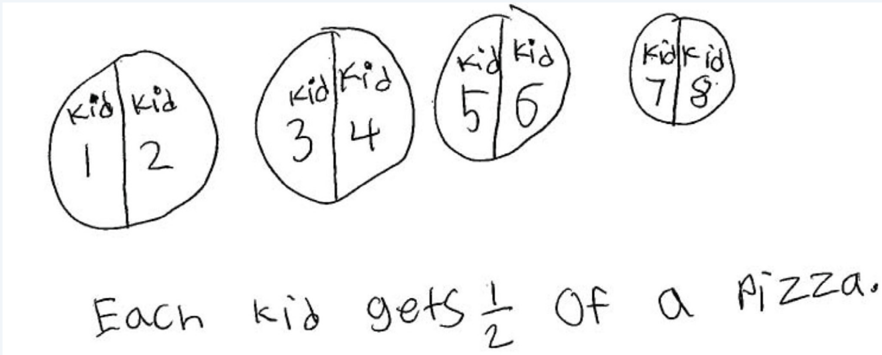
Building on the activities in Strategy 1 above, teachers can extend students' understanding to ordering fractions and identifying equivalent fractions using similar story problems involving a group of people sharing objects when students continue to use manipulatives and/or drawings to help make sense of the problem. However, teachers extend the scenarios to require fraction comparisons or identification of equivalent fractions by focusing on different aspects of students' solution strategies.

To help students better understand relative size, for example, teachers can gradually increase the number of people who will be sharing and have students compare the relative amounts that each person receives. In this, teachers should guide students to observe that increasing the number of people reduces the amount that each person receives and vice versa. Teachers should help students link this idea to the formal fraction names they have been using to identify the quantities, guiding them to use these names to discuss the results of their solutions (e.g., $1/3$ of an object is greater than $1/4$ of that same object).

In these scenarios, teachers should include both of the following approaches:

- Partition objects into larger and smaller pieces. Teachers should guide students to think about different ways to solve the same problem. This might involve partitioning the object(s) into smaller or larger pieces while ensuring that each person receives an equal amount. In this way, teachers can help students see that groups of the smaller pieces can be combined to be the same size as a larger piece. For example, as shown below, a student may solve the problem of eight people sharing four pizzas by cutting each pizza in half and giving each person half of a pizza. This problem could also be solved by cutting each pizza into quarters and giving each person two quarters. Thus, one half is the same as, or equivalent to, two quarters (i.e., $1/2 = 2/4$).
- Partition the number of sharers and the number of items. In this approach, teachers help students build the idea of fraction equivalence by partitioning both the number of sharers and the number of objects. For example, in the problem shared previously, a student may partition each pizza into eighths and give each person four pieces or four eighths. The teacher could extend the problem to have the student compare what would happen if the people were split into two tables so that each table of four had to share two pizzas...or that people were split into four tables. In each case, each person still receives one-half of the pizza. Every situation is equivalent.

Example of Student Work for Sharing Four Pizzas Among Eight Students



Note. Taken from Figure 3 on page 16 of the practice guide (Siegler et al., 2010).

Teachers can also help students develop their understanding of equivalent fractions by using "missing value "problems. Here, students need to determine the number of objects they would need to result in an equivalent share. For example, the practice guide (p. 17) presents the following: "If six students share eight oranges at one table, how many oranges are needed at a table of three students to ensure each student receives the same amount?" (Note: Further scenarios and examples of students" thinking are also provided on p. 17 of the practice guide.)

3. Build on students' informal understanding to develop a more advanced understanding of proportional reasoning concepts. Begin with activities that involve similar proportions, and progress to activities that involve ordering different proportions.

Instructional strategies from the examples:

- Use proportional relations, covariation, and/or patterns to help develop students' proportional thinking.
- Use scenarios that can't directly be quantified to help students think about proportional relations outside of a full focus on numbers.

To help build students' understanding of proportional reasoning, teachers might use a story like Goldilocks and the Three Bears that asks students to think about the proportional relationships between pairs of objects without providing specific numbers (e.g., Papa Bear goes with the large chair, Baby Bear goes with the small bed).

Here are some examples of different relations teachers might use:

- **Proportional relations.** As with Goldilocks and the Three Bears, teachers present students with basic proportional relations that are not quantified (e.g., do not present actual values). For example, how many students would it take to balance a seesaw with one, two, or three adults on the other side?
- **Covariations.** These types of scenarios involve one quantity increasing while another is also increasing (e.g., the age of a student and the height of the student).
- **Patterns.** Teachers can use simple repeating patterns to develop a scenario (e.g., RRYRRYRRYRRY), then discuss with students how many red beads there are for every yellow bead. Students can then change the pattern to a different ratio or extend the pattern to solidify proportional reasoning.

Potential Roadblocks

Students are unable to draw equal-sized parts.

Suggested Approach. Teachers should let students know that it's okay to draw parts that aren't exactly equal as long as they recognize that the parts are supposed to be equal when they work on solving the problem.

Students do not share all the items (non-exhaustive sharing) or do not create equal shares.

Suggested Approach. Although teachers are building on students' intuitive understanding of sharing equally, students will still make mistakes, such as not sharing all the objects, especially when the problem involves partitioning one or more of the objects. Teachers should focus on helping students understand that sharing involves using all the objects (e.g., letting students know that each person needs to get all they possibly can).

If students do not create equal shares, teachers can reframe by reminding students that each person needs to receive the same amount (e.g., things need to be shared fairly). Equal sharing like this helps lay the foundation for students' understanding of equivalent fractions and equivalent magnitude difference (e.g., that a difference of 1/2 is the same whether it's between 0 and 1/2 or 23 and 23 1/2).

When creating equal shares, students do not distinguish between the number of things shared and the quantity shared.

Suggested Approach. Limited experience often leads students to make the mistake of giving each person the same number of items rather than thinking about the amount, especially when items shared may be of different sizes. Teachers can use color cues to help students distinguish between the number and the amount. For example, if five small and five large food items are being shared between two dogs, the teacher may color the large items green and the small items red, give all five green items to one dog and all five red items to the other, then ask students if the dogs received the same amount of food.

Content Area	Level of Evidence	Grades
Math	Moderate	K, 1st, 2nd, 3rd, 4th, 5th, 6th, 7th, 8th

RECOMMENDATION 2

Help students recognize that fractions are numbers and that they expand the number system beyond whole numbers. Use number lines as a central representational tool in teaching this and other fraction concepts from the early grades onward.

Initial fraction learning often starts with the concept of fractions as part of a whole, but that understanding does not express the fact that fractions are numbers with magnitudes that can be compared (e.g., ordered, considered equivalent). Missing this understanding is often at the root of many misconceptions about fractions (e.g., adding to fractions by adding the numerators, then adding the denominators; not seeing fractions as numbers or units of measurement). Using number lines is an effective way to help develop an understanding of fractions as numbers with magnitudes because they can provide a clear picture of the magnitude of fractions, the relationship between fractions and whole numbers, and the relationship between fractions, decimals, and percentages. Number lines also provide a foundation for students' number sense with fractions and a way to visualize negative fractions.

Strategies

1. Use measurement activities and number lines to help students understand that fractions are numbers, with all the properties that numbers share.

Instructional strategies from the examples:

- Use measurement activities to develop the idea that fractions allow for more precise measurement than whole numbers.
- Present situations in which fractions are used to solve problems that cannot be solved only using whole numbers.
- Show students the various measurement lines on a measuring cup and convey the importance of fractions in describing quantities.

Helping students view fractions as numbers opens the door for them to use fractions to measure quantities and understand that using fractions allows for more precise measurement of objects. Fractions can also help solve certain problems that cannot be solved using only whole numbers (e.g., describing the amount of sugar needed for a recipe that asks for more than 1 cup but less than 2 cups). Fraction strips are length models that teachers can use to reinforce the idea of fractions representing quantities.

These fraction strips are also known as fraction strip drawings, strip diagrams, bar strip diagrams, and tape diagrams.

Example of Measurement Activities with Fraction Strips

Using Fraction Strips to Measure an Object

Teachers can use fraction strips as the basis for measurement activities to reinforce the concept that fractions are numbers that represent quantities.

To start, students can take a strip of card stock or construction paper that represents the initial unit of measure (i.e., a whole) and use that strip to measure objects in the classroom (desk, chalkboard, book, etc.). When the length of an object is not equal to a whole number of strips, teachers can provide students with strips that represent fractional amounts of the original strip. For example, a student might use three whole strips and a half strip to measure a desk.

Teachers should emphasize that fraction strips represent different units of measure and should have students measure the same object first using only whole strips and then using a fractional strip. Teachers should discuss how the length of the object remains the same but how different units of measure allow for better precision in describing it. Students should realize that the size of the subsequently presented fraction strip is defined by the size of the original strip (i.e., a half strip is equal to one half of the length of the original strip).

Note. Taken from Example 1 on page 21 in the practice guide (Siegler et al., 2010).

2. Provide opportunities for students to locate and compare fractions on number lines.

Instructional strategies from the examples:

- Provide opportunities for students to locate and compare fractions on number lines, beginning with fully labeled number lines and gradually reducing the labels.
- Use number lines to compare various fractions to whole numbers greater than one.
- Encourage students to think about the distance between two fractions.

Activities involving comparing fractions should include a variety of fraction forms, such as proper fractions, improper fractions, mixed numbers, whole numbers, decimals, and percentages. Teachers should start by providing number lines with fractional parts already marked and move to numbers lines that are more minimally labeled. This helps avoid student errors in partitioning accurately and also allows for location and comparison of fractions whose locations are indicated (e.g., $3/8$ and $5/8$ on a ruler), as well as fractions whose denominators are a factor of the indicated fractions (e.g., $1/4$ and $3/4$) and fractions between those indicated (e.g., $1/7$ and $3/5$).

Activities should also involve comparison with whole numbers, include fractions equivalent to whole numbers (e.g., locating 1 and $8/8$) and comparing fractions of various sizes to whole numbers greater than one (e.g., locating $10/3$ on a number line, with 0 at the left end and 5 at the right end). To assist students in comparing fractions with different denominators, teachers can label number lines with one fractional-unit sequence above and a different sequence below.

One activity teachers can use with the whole class involves drawing a number line on the board and having students estimate and mark where different fractions fall. The teacher can guide students' discussion about fractions yet to be placed, focusing on maintaining the correct order and encouraging thinking about the distance between fractions. Again, including a variety of fraction types, and also decimals and percentages, helps solidify student understanding.

Example of Introducing Fractions on a Number Line

To illustrate the location of $3/5$ on a 0-to-5 number line, a teacher might first mark and label the location of 1 and then divide the space between each whole number into five equal-size parts. After this, they might add the labels $0/5$, $1/5$, $2/5$, $3/5$, $4/5$, and $5/5$ in the 0–1 portion of the number line and highlight the location of $3/5$. Displaying whole numbers as fractions (e.g., $5/5$) allows teachers to discuss what it means to describe whole numbers in terms of fractions and to clarify that whole numbers are fractions, too.

Note. Adapted from Example 2 on page 22 in the practice guide (Siegler et al., 2010).

3. Use number lines to improve students' understanding of fraction equivalence, fraction density (the concept that there are an infinite number of fractions between any two fractions), and negative fractions.

Instructional strategies from the examples:

- Use number lines to:
  - Illustrate that equivalent fractions describe the same magnitude.
  - Illustrate that an infinite number of fractions exist between any two other fractions.
  - Introduce negative fractions.
  - Convey the symmetry of positive and negative fractions about zero.

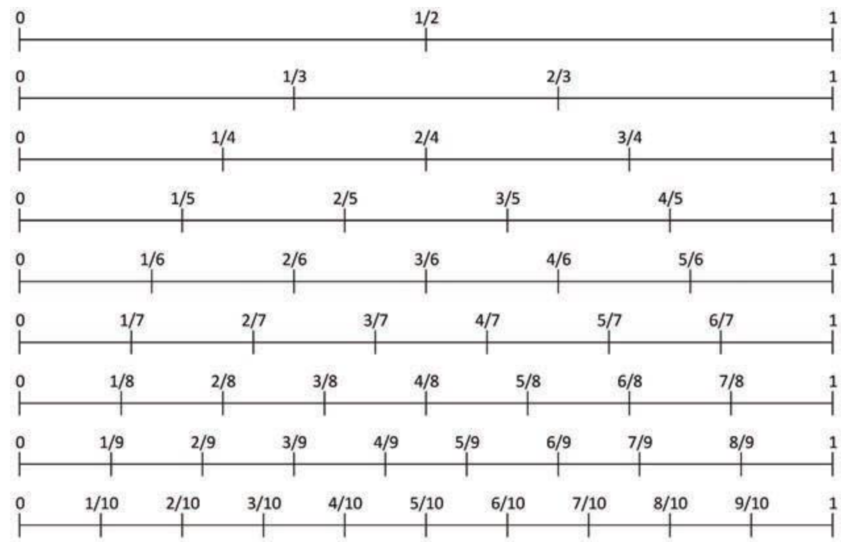
Number lines are also useful tools for helping students think about and identify equivalent fractions, negative fractions, and fraction density.

To illustrate that equivalent fractions have the same magnitude, a teacher might provide a number line labeled with one fractional sequence on top and a different fractional sequence on the bottom. Students can locate fractions on each of the sequences and discuss how they are equivalent when they are in the same location on the number line. Discussions about equivalent fractions should build the measurement activities from the previous strategies above, using rulers and fraction strips to reinforce.

Another challenging concept for students to understand is that an infinite number of fractions exist between any two other fractions on the number line. This is an important difference between fractions and whole numbers. To help students understand this concept, teachers can ask them to successively partition a portion of the number line into smaller and smaller unit fractions (e.g., repeatedly dividing a whole number segment in half). Similarly, teachers should use decimals and percentages to represent this concept.

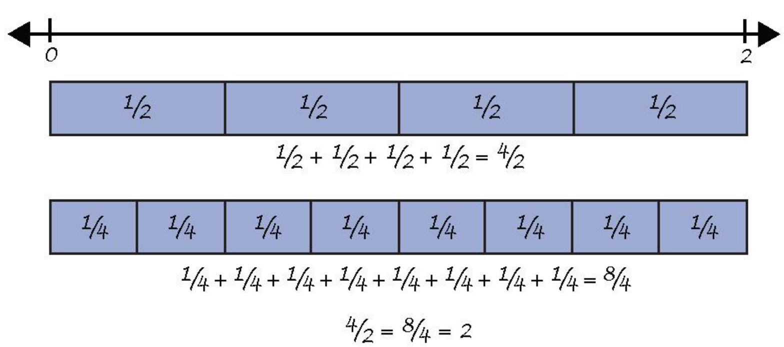
The number line provides an excellent visual representation of fractions both greater than and less than zero. By providing number lines that include marks and labels for zero, several positive fractions, and several negative fractions with the same absolute values as the positive fractions, teachers can help convey the symmetry about zero of positive and negative fractions.

Example of Using Numbers Lines to Show Equivalence of Fractions



Note. Taken from Example 4 on page 23 in the practice guide (Siegler et al., 2010).

Example of Using Fraction Strips to Demonstrate Equivalent Fractions



Note. Taken from Example 5 on page 24 in the practice guide (Siegler et al., 2010).

4. Help students understand that fractions can be represented as common fractions, decimals, and percentages, and develop students' ability to translate among these forms.

Instructional strategies from the examples:

- Use a number line with common fractions listed above with equivalent decimals and percentages below.

Number lines can provide a useful tool in helping students develop a broad view of fractions as numbers, including understanding that fractions can be represented as decimals and percentages, as well as common fractions. Fractions, decimals, and percentages are just different ways of representing the same number. By using a number line with common fractions listed above and decimals or percentages below, students can locate and compare fractions, decimals, and percentages on the same number line.

Potential Roadblocks

Students try to partition the number line into fourths by drawing four hash marks rather than three, or they treat the whole number line as the unit.

Suggested Approach. Teachers may need to teach students that "fourths" is represented as four equal segments between two whole numbers, demonstrating that three equally spaced hatch marks divides the space into four equal parts. Later, guide students to generalize this rule to know that to divide a portion of the number line between two whole numbers into $1/n$ units involves drawing $n-1$ hatch marks between the two whole numbers.

When students locate fractions on the number line, they treat the numbers in the fraction as whole numbers (e.g., placing 3/4 between 3 and 4).

Suggested Approach. This error is based in students' misconception where they try to apply whole number understanding to fractions. Teachers can help address this misconception through using number lines and providing contrasting cases where students locate, for example, 3 and 4 on a 0-to-4 number line, then locate $3/4$ between 0 and 1. Teachers should follow this activity with discussion about why each is placed where it is on the number line.

Students have difficulty understanding that two equivalent fractions are the same point on a number line.

Suggested Approach. Teachers can show students one set of labels above the number line and another below the number line. For example, halves could be marked above and eighths below. Teachers can then point to the equivalent positions of $1/2$ and $4/8$ or 1, $2/2$, and $8/8$, and so on. Another approach would be for students to create a number line showing $1/2$ and another number line showing $4/8$, and then compare the two. Teachers can line up the two number lines and lead a discussion about equivalent fractions.

The curriculum materials used by my school district focus on part-whole representations and do not use the number line as a key representational tool for fraction concepts and operations.

Suggested Approach. Understanding that fractions can represent part of a whole is only one use. Using measurement and number lines provides an additional context to help students build a rich understanding of fractions. Teachers can also use manipulatives designed to represent parts of a whole for measurement purposes; when doing this, teachers should make sure students aren't simply counting pieces represented by the numerator and denominator but are instead seeing the fraction as a single quantity. Using number lines that are unmarked between endpoints can help, because they do not provide objects to count. Additionally, seek out other textbooks and resources that may provide examples of using fractions as measures of quantity.

Content Area	Level of Evidence	Grades
Math	Moderate	K, 1st, 2nd, 3rd, 4th, 5th, 6th, 7th, 8th

RECOMMENDATION 3

Help students understand why procedures for computations with fractions make sense.

Building conceptual understanding is foundational for students' proficient use of computational procedures, but procedures with fractions are often taught without helping students understand how or why the procedures work. Teachers should help students build both procedural fluency and conceptual understanding by explaining to students how the computations procedures they are presenting transform fractions in meaningful ways. Using practices such as visual representations, estimation, and setting problems in real-world contexts help reinforce students' conceptual understanding.

Strategies

1. Use area models, number lines, and other visual representations to improve students' understanding of formal computational procedures.

Instructional strategies from the examples:

- Use visual representations and manipulatives to build insight into concepts underlying computational procedures and the reasons why these procedures work.
- Model the division of fractions with representations such as ribbons or a number line.
- Consider the advantages and disadvantages of different representations for teaching procedures for computing with fractions and whether a representation can be used with different types of fractions.

Teachers can use visual representations and manipulatives (e.g., number lines and area models) to help students understand basic concepts underlying conceptual procedures with fractions and the reasons why the procedures work.

Here are some examples of how teachers can use representations of fractions in this way:

- **Find a common denominator when adding and subtracting fractions.** A common misunderstanding with students is that you can add fractions by adding the numerators and adding the denominators. Teachers can use a variety of representations to provide visual cues to help students see the need for common denominators.

  For example, if students are given two circle fractions representing $1/2$ and $1/3$, the teacher can show how converting both into sixths provides a common denominator that applies to both fractions, allowing the student to add the two fractions together (see Example 1 below). Teachers should build on this understanding by discussing with students why multiplying the denominators results in a common denominator that applies to both original fractions.

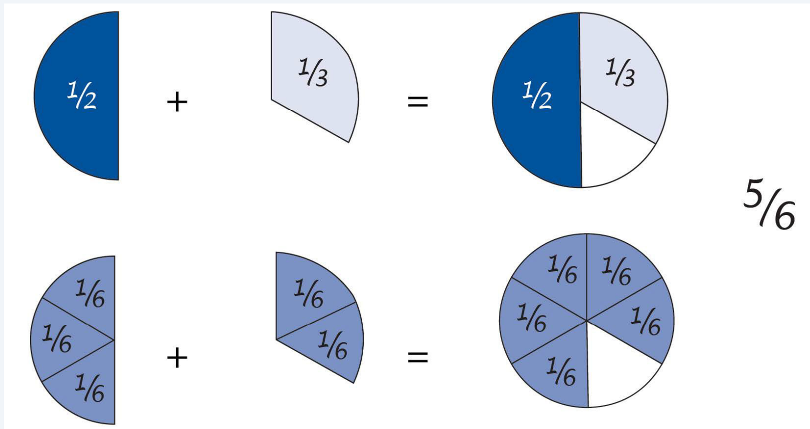
- **Redefine the unit when multiplying fractions.** Using concrete or pictorial representations can help students visualize that multiplying two fractions equates to finding a fraction of a fraction. Area models, for example, provide good examples, as shown in Example 2 below. The approach in Example 2 demonstrates how the initial unit is the full cake, treating the cake as the whole and dividing it into thirds, then moving to treat the portion shaded in the first step as the whole and dividing it into fourths.

- **Divide a number into fractional parts.** Although dividing fractions may look different on the surface, it is conceptually similar to dividing whole numbers. That is, students can think about how many times the divisor goes into the dividend. Teachers can use representations such as number lines and ribbons to model division of fractions. For example, in Example 3 below, ribbons are used to model $1/2 \div 1/4$, helping students think in terms of "How many $1/4$s are there in $1/2$?"

When selecting representation, teachers should consider the advantages and disadvantages of each to ensure that the representation adequately reflects the computational process students are expected to learn, allowing them to make connections between the representation and the computation. Teachers should also think about the facility of the representation in working with different types of fractions (e.g., proper fractions, mixed numbers, improper fractions, and negative fractions), as well as with other mathematical concepts (e.g., representing decimals with base-10 blocks, using 100 grids where one square is seen as $1/100$ of the whole).

Example 1. Using Fraction Circles for Addition

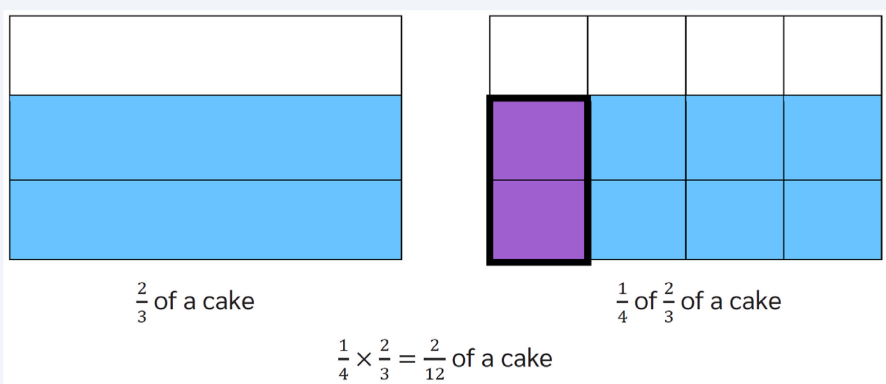
Adding 1/2+1/3 using fraction circles:



Note. Taken from Figure 6 on page 28 in the practice guide (Siegler et al., 2010).

Example 2. Redefining the Unit when Multiplying Fractions

Lori is icing a cake. She knows that 1 cup of icing will cover 2/3 of a cake.
How much cake can she cover with 1/4 cup of icing?



Note. Taken from Figure 7 on page 29 in the practice guide (Siegler et al., 2010).

Example 3. Using Ribbons to Model Division with Fractions

Students use ribbons to solve $^1/_2 \div {}^1/_4$.

Step 1. Divide a ribbon into fourths.

$^1/_4$	$^1/_4$	$^1/_4$	$^1/_4$

Step 2. Divide a ribbon of the same length into halves.

$^1/_2$	$^1/_2$

Step 3. Compare the two ribbons to find out how many fourths of a ribbon are the same length as (that is, "can fit it into") one half.

$^1/_4$	$^1/_4$
⇩	⇩
$^1/_2$	

Two-fourths are the same length as ("fit into") one half of the ribbon.

So ½ + ¼ + ¼ = 2.

Note. Taken from Figure 8 on page 30 in the practice guide (Siegler et al., 2010).

2. Provide opportunities for students to use estimation to predict or judge the reasonableness of answers to problems involving computation with fractions.

Instructional strategies from the examples:

- Provide opportunities for students to estimate the solutions to problems.
- Discuss whether and why students' solutions to specific problems are reasonable to help improve students' estimation skills.
- Explicitly teach estimation strategies.

In conjunction with teacher computation with fractions, providing students with opportunities to estimate their solutions helps build their reasoning skills and focus on the meaning of the computational procedures. Teachers should ask students to provide an initial estimation and explain their thinking before having them actually compute the answer to help students judge the reasonableness of the answers they compute. Teachers can help improve both students' estimation skills and their judgment of the reasonableness of computed answers by having students discuss the strategies they used to determine their estimate and compare their initial estimates to the solutions they've computed.

Teachers should provide opportunities for students to estimate solutions for problems in which the solution cannot be easily computed. Additionally, explicitly teaching effective strategies for estimation can help maximize the value that estimation has in deepening students' understanding of fraction computation.

Example 4. Discussing Estimation Strategies and Reasonableness of Solution

Problem
$1/2+1/5$
Solution
STUDENT: I estimate that $1/2+1/5$ is more than $1/2$ but less than $3/4$. **TEACHER:** Why do you think that? **STUDENT:** Well, I know that $1/2+1/4=3/4$. $1/5$ is less than $1/4$, so the answer for $1/2+1/5$ will be less than $3/4$. **TEACHER:** So, let's compute the sum and see. Student computes the sum and arrives at a solution of $2/7$ because they incorrectly add the numerators and denominators. **TEACHER:** Do you think that $2/7$ is a reasonable solution for $1/2+1/5$? **STUDENT:** It can't be, because that's less than $1/2$, and I know the answer has to be bigger than $1/2$. Teacher follows up with further discussion of the procedure the student used, helping them identify the error made and guiding them to understand the correct procedure and solution.

Note. Adapted from the example on page 31 in the practice guide (Siegler et al., 2010).

Example 5. Strategies for Estimating with Fractions

Strengthening estimation skills can develop students' understanding of computational procedures.

Benchmarks. One way to estimate is through benchmarks—numbers that serve as reference points for estimating the value of a fraction. The numbers 0, $1/2$, and 1 are useful benchmarks because students generally feel comfortable with them. Students can consider whether a fraction is closest to 0, $1/2$, or 1. For example, when adding $7/8$ and $3/7$, students may reason that $7/8$ is close to 1, and $3/7$ is close to $1/2$, so the answer will be close to $1/12$. Further, if dividing 5 by $5/6$, students might reason that $5/6$ is close to 1, and 5 divided by 1 is 5, so the solution must be a little more than 5.

Relative size of unit fractions. A useful approach to estimating is for students to consider the size of unit fractions. To do this, students must first understand that the size of a fractional part decreases as the denominator increases. For example, to estimate the answer to $9/10+1/8$, beginning students can be encouraged to reason that $9/10$ is almost 1, that $1/8$ is close to $1/10$, and that therefore the answer will be about 1. More advanced students can be encouraged to reason that $9/10$ is only $1/10$ away from 1, that $1/8$ is slightly larger than $1/10$, and therefore the solution will be slightly more than 1. The principle can and should be generalized beyond unit fractions once it is understood in that context. Key dimensions for generalization include estimating results of operations involving non-unit fractions (e.g., $3/4\div2/3$, improper fractions ($7/3\div3/4$), and decimals ($0.8\div0.33$).

Placement of Decimal Point. A common error when multiplying decimals, such as 0.8×0.9 or 2.3×8.7, is to misplace the decimal. Encouraging students to estimate the answer first can reduce such confusion. For example, realizing that 0.8 and 0.9 are both less than 1 but fairly close to it can help students realize that answers such as 0.072 and 7.2 must be incorrect.

Note. Taken from Example 3 on page 31 in the practice guide (Siegler et al., 2010).

3. Address common misconceptions regarding computational procedures with fractions.

Instructional strategies from the examples:

- Identify students who are operating with misconceptions about fractions, discuss the misconceptions with them, and make clear both why the misconceptions lead to incorrect answers and why correct procedures lead to correct answers.

Students' understanding of computational procedures with fractions can often be limited by misconceptions they have about fractions. Once teachers have identified misconceptions students have, they can present these in discussions about why some procedures used by students result in correct answers, while others do not. Some common misconceptions are described here:

- **Believing that fractions' numerators and denominators can be treated as separate whole numbers.** A common mistake students make when adding or subtracting fractions is to try to add/subtract the numerators, then do the same with the denominators. This error is rooted in students applying their knowledge of whole numbers to fractions and not understanding that the denominator defines the size of the fractional part, while the numerator represents the number of parts of that size. Additionally, the fact that this approach works for the multiplication of fractions adds support for the misconception.

  To help students overcome this misconception, teachers should present meaningful problems to students, setting problems within real-world contexts that are relevant for their students. For example, page 32 in the practice guide presents the following: "If you have $3/4$ of an orange left and give $1/3$ of it to a friend, what fraction of the original orange do you have left?" In computing a solution, students should recognize that $34-13=21$ can't be correct because they can't come up with 2 oranges if they only started with $3/4$ of an orange in the first place. In using meaningful problems like this, teachers can help lay a foundation for students to think deeply about why treating the numerators and denominators as separate whole numbers is inappropriate, opening the door for discussion about appropriate procedures.

- **Failing to find a common denominator when adding or subtracting fractions with unlike denominators.** When adding or subtracting fractions with unlike denominators, a common mistake is that students just insert the larger denominator into the solution rather than convert both fractions to equivalent fractions with a common denominator. For example, when calculating $4/5+4/10$, students may find an incorrect solution of $8/10$. This misconception is rooted in students not understanding that different denominators represent different-sized unit fractions.

A similar error of not converting the numerator of a fraction when changing the denominator is also rooted in this misconception (e.g., $2/3+2/6$ incorrectly becomes $2/6+2/6$). Teachers can use visual representations that show equivalent fractions (e.g., number lines, fraction strips) to help students see both the need for common denominators (i.e., equal sizes of unit-fraction parts) and appropriate changes to numerators.

- **Believing that only whole numbers need to be manipulated in computations with fractions greater than one.** Mixed numbers provide an additional challenge for students with misconceptions about fractions. Often, in addition/subtraction problems with mixed numbers, students may ignore the fractional part and work only with the whole numbers (e.g., $5\ 3/5 + 2\ 1/7 = 7$). Errors such as these may result from students ignoring the part of the problem they do not understand, misunderstanding the meaning of mixed numbers, or assuming that such problems simply have no solution.

  Related to this, students may think that the whole-number portion of a mixed number has the same denominator as the fraction in a problem. This misconception might lead students to incorrectly translate the problem $5-2/3$ into $5/3-2/3$. This misconception might also lead students to want to add the whole number to the numerator in more complex problems. Page 32 in the practice guide presents the following example of this type of error: $3\ 1/3\times6/7=(3/3+1/3)\times6/7=4/3\times6/7=24/21$.

  To help overcome this misconception, teachers should help students understand the relationship between mixed numbers and improper fractions. Understanding how to translate each into the other is crucial for working with these fractions.

- **Treating the denominator the same in fraction addition and multiplication problems.** Misconceptions about fractions also lead to errors like students mixing up procedures for adding and multiplying fractions. This often results in computational errors, such as students leaving the denominator unchanged in fraction multiplication problems involving fractions with common denominators (e.g., $2/5\times3/5=6/5$). Errors such as this might also result from the fact that students see more fraction addition problems than multiplication problems, leading them to generalize procedures from fraction addition incorrectly to multiplication.

  Teachers can address this misconception by helping students focus on the conceptual basis for fraction multiplication. For example, the product $1/2\times1/2$ can be thought of as "half of a half" and students can visualize that half of one half would equal one fourth (i.e., $1/2\times1/2=1/4$). Additionally, teachers can help students think about the fact that multiplying by a fraction less than 1 results in a product that is smaller.

- **Failing to understand the invert-and-multiply procedure for solving fraction division problems.** When students lack a conceptual understanding, they often misapply procedures like "invert and multiply" (see Example 6 below). Errors like this often reflect a lack in conceptual understanding regarding why the invert-and-multiply procedure translates a multistep calculation into a more efficient way to calculate the correct quotient.

Example 6. Common Student Errors with the Invert-and-Multiply Procedure

Problem
$2/3 \div 4/5$
Solution
Error 1—not inverting either fraction. Students may understand the procedure involves multiplying the two fractions but forget the "invert" part. $2/3 \div 4/5 = 8/15$
Error 2—inverting the wrong fraction. $2/3 \div 4/5 = 3/2 \times 4/5 = 12/10$
Error 3—inverting both fractions. $2/3 \div 4/5 = 3/2 \times 5/4 = 15/8$

Note. Taken from the examples on page 33 in the practice guide (Siegler et al., 2010).

Teachers should ensure students understand the multistep calculation that is the basis of this procedure. The invert-and-multiply procedure is based in two mathematical concepts: (1) multiplying a number by its reciprocal results in a product of 1, and (2) dividing any number by 1 leaves the number unchanged. Teachers can show students how these connect to the invert-and-multiply procedure as follows for the problem $2/3 \div 4/5$:

- Multiplying both the dividend ($2/3$) and divisor ($4/5$) by the reciprocal of the divisor yields ($2/3 \times 5/4$)÷($4/5 \times 5/4$).
- Multiplying the original divisor ($4/5$) by its reciprocal ($5/4$) produces a divisor of 1, which results in $2/3 \times 5/4 \div 1$, which yields $2/3 \times 5/4$.
- Thus, the invert-and-multiply procedure, multiplying $2/3 \times 5/4$, provides the solution.

4. Present real-world contexts with plausible numbers for problems that involve computing with fractions.

Instructional strategies from the examples:

- Use real-world contexts that provide meaning to the fractions involved in the problem.
- Tailor problems around details that are familiar and meaningful to the students; gather ideas from them.
- Make connections between a real-world problem and the fraction notation used to represent it.

Teachers should present students with problems that use plausible numbers in real-world contexts. The contexts should also provide meaning to both the fraction quantities involved and the computational procedures used to find a solution. Setting problems in real-world measurement contexts (e.g., using rulers, ribbons, or measuring tapes) and using food in the problem can help.

When including food items, teachers should use both discrete (e.g., boxes of candy, apples) and continuous items (e.g., pizza, candy bars). A good source of ideas for relevant contexts are the students themselves, as they will help tailor problems around details familiar and meaningful to them (e.g., school events, field trips, activities in other subjects).

When providing real-world problems and contexts, teachers should try to help students connect the problem with the fraction notation used to represent it. At times, students can correctly solve a problem presented in a real-world context but struggle to solve the same problem when presented in formal notation. Teachers should continue to connect the notation back to the real-world story problem as students work through the solution.

Potential Roadblocks

Students make computational errors (e.g., adding fractions without finding a common denominator) when using certain pictorial and concrete object representations to solve problems that involve computation with fractions.

Suggested Approach. Teachers should carefully choose representations that map easily and most directly to the fraction computation they are teaching (e.g., demonstrating the need for similar units when adding fractions), as use of some representations can actually reinforce misconceptions. To reinforce the need for common unit fractions, teachers should consider using representations that hold units constant (e.g., measuring tapes).

When encouraged to estimate a solution, students still focus on solving the problem via a computational algorithm rather than estimating it.

Suggested Approach. Teachers should present estimation as a preliminary tool for helping anticipate the relative size and appropriateness of a solution, not a shortcut to an answer. In this, teachers should help students focus on the reasoning needed to estimate a solution. Posing problems that are not quickly solvable with mental computation (e.g., $5/9+3/7$ instead of $5/8+3/8$) will help avoid this roadblock.

Content Area	Level of Evidence	Grades
Math	Minimal	K, 1st, 2nd, 3rd, 4th, 5th, 6th, 7th, 8th

RECOMMENDATION 4

Develop students' conceptual understanding of strategies for solving ratio, rate, and proportion problems before exposing them to cross-multiplication as a procedure to use to solve such problems.

"Thinking proportionally" means that students understand the multiplicative relationship between two quantities and is a critical skill students must develop in preparation for success in advanced mathematics. Ratios, rates, and proportions are contexts that require understanding of multiplicative relationships and lead into the cross-multiplication algorithm. Teachers should develop students' proportional reasoning skills before teaching the cross-multiplication algorithm using a progression of problems that help build their informal reasoning strategies. Teachers should also make sure to return to the informal reasoning strategies after teaching the cross-multiplication algorithm, demonstrating that the algorithm and the informal reasoning students use lead to the same answers. As a caution, studies involving many types of problem-solving have demonstrated that students often learn a strategy to solve a problem in one context but fail to successfully generalize to other contexts.

Strategies

1. Develop students' understanding of proportional relations before teaching computational procedures that are conceptually difficult to understand (e.g., cross multiplication). Build on students' developing strategies for solving ratio, rate, and proportion problems.

Instructional strategies from the examples:

- Use a progression of problems that builds on students' developing strategies for proportional reasoning.
- Encourage students to apply their own strategies, discuss the various strategies' strengths and weaknesses, and help them understand why a problem's solution is correct.

Teachers should provide opportunities for students to solve ratio, rate, and proportion problems, building students' strategies for proportional reasoning, before teaching the cross-multiplication algorithm. While encouraging students to apply their own strategies, teachers should also introduce ways to solve these problems if students struggle with generating their own strategies. In this, teachers should discuss the various strengths and weaknesses of strategies and help students understand why the resulting solution is correct.

A possible progression teachers might use to guide students from informal proportional reasoning strategies to the cross-multiplication algorithm might look like this (see Example 1):

- **Buildup Strategy.** Initially pose story problems that allow students to use a buildup strategy where they repeatedly add the numbers within one ratio to solve a problem. In these problems, teachers should ensure the numbers in the ratios are integrally related so that one can be generated by repeatedly adding the numbers in the other (e.g., $2{:}3$ and $10{:}15$). These should begin with problems involving smaller numbers to allow students to build their understanding, then progress to problems with larger numbers to demonstrate how time-consuming it can be to repeatedly add to these large numbers. This will help students recognize the value of using multiplication and division.
- **Unit Ratio Strategy.** Next, teachers can present problems that cannot be easily solved through repeated addition or through multiplying/dividing by a single integer (e.g., $x/6=3/9$). Solving these types of problems involves reducing the known ratio to a form with a numerator of 1, then determining the multiplicative relationship between the denominator in the new unit ratio and the ratio with the unknown value. This relationship can then be used to solve for the unknown value. This strategy can also be used in problems where the solution is not a whole number and bridge into helping students see the value in the cross-multiplication algorithm.
- **Cross-Multiplication.** Building on students' understanding of the unit ratio strategy, teachers can present problems that do not involve integral relations or that use ratios that cannot be easily reduced to unit fractions. This will help students see the advantages of using a strategy that can help solve problems, regardless of the numbers involved. Teachers should continue to help students make connections by having them return to previous strategies to see that cross-multiplication results in the same answer and discussing why this is the case.

Teachers should continue to present problems that can be solved easily through informal reasoning and mental mathematics, as well as those more easily solved with cross-multiplication. In this, teachers can discuss with students how to anticipate which approach might be easiest for a given problem.

Example 1. Problems Encouraging Specific Strategies

Buildup Strategy
Sample problem. If Steve can purchase 3 baseball cards for \$2, how many baseball cards can he purchase with \$10?   **Solution approach.** Students can build up to the unknown quantity by starting with 3 cards for \$2, and repeatedly adding 3 more cards and \$2, thus obtaining 6 cards for \$4, 9 cards for \$6, 12 cards for \$8, and, finally, 15 cards for \$10.

Unit Ratio Strategy
Sample problem. Yukari bought 6 balloons for \$24. How much will it cost to buy 5 balloons?   **Solution approach.** Students might figure out that if 6 balloons costs \$24, then 1 balloon costs \$4. This strategy can later be generalized to one in which eliminating all common factors from the numerator and denominator of the known fraction does not result in a unit fraction (e.g., a problem such as $6/15=x/10$, in which reducing $6/15$ results in $2/5$).

Cross-Multiplication
Sample problem. Luis usually walks the 1.5 miles to his school in 25 minutes. However, one of the streets on his usual path is being repaired today, so he needs to take a 1.7-mile route. If he walks at his usual speed, how much time will it take him to get to his school?   **Solution approach.** This problem can be solved in two stages. First, because Luis is walking at his “usual speed,” students know that $1.5/25=1.7x$. Then, the equation may be most easily solved using cross-multiplication. Multiplying 25 and 1.7 and dividing the product by 1.5 yields the answer of 28.33 minutes, or 28 minutes and 20 seconds. It would take Luis 28 minutes and 20 seconds to reach school using the route he took today.

Note. Taken from Example 4 on page 38 of the practice guide (Siegler et al., 2010).

Example 2. Why Cross-Multiplication Works

Teachers can explain why the cross-multiplication procedure works by starting with two equal fractions, such as $4/6=6/9$. The goal is to show that when two equal fractions are converted into fractions with the same denominator, their numerators also are equivalent. The following steps help demonstrate why the procedure works.

Step 1. Start with two equal fractions, for example: $4/6=6/9$.

Step 2. Find a common denominator using each of the two denominators.

a. First, multiply $4/6$ by $9/9$, which is the same as multiplying $4/6$ by 1.

b. Next, multiply $6/9$ by $6/6$, which is the same as multiplying $6/9$ by 1.

Step 3. Calculate the result: $(4\times9)(6\times9)=(6\times6)(9\times6)$

Step 4. Check that the denominators are equal. If two equal fractions have the same denominator, then the numerators of the two equal fractions must be equal as well, so $4 \times 9 = 6 \times 6$.

Note that in this problem, $4 \times 9 = 6 \times 6$ is an instance of $(a \times d = b \times c)$.

As a result, students can see that the original proportion $4/6 = 6/9$ can be solved using cross-multiplication, $4 \times 9 = 6 \times 6$, as a procedure to create equivalent ratios efficiently.

Note. Taken from Example 5 on page 39 of the practice guide (Siegler et al., 2010).

2. Encourage students to use visual representations to solve ratio, rate, and proportion problems.

Instructional strategies from the examples:

- Select representations that are likely to elicit insight into a particular aspect of ratio, rate, and proportion concepts.
- Encourage students to create their own representations.

Teachers should encourage students to use visual representations when solving ratio, rate, and proportion problems and select representations that highlight specific concepts within the problem.

For example, teachers can use a ratio table to represent the relations in a proportion problem and provide specific reference to highlight how multiplication leads to the same solution as the buildup strategy (see Example 3). In addition, teachers can use ratio tables to help students explore different aspects of proportional relationship, such as multiplicative relationships within and between ratios (see Example 3).

Teachers should encourage students to develop their own representations and not always provide them. With ratios, rates, and proportions, students initially tend to use tabular or other systematic record-keeping formats. Through formal instruction and exposure, teachers can introduce other representations and encourage students to use these in future problems.

Example 3. Ratio Table for a Proportion Problem

Problem

How many cups of flour are needed for 32 people when a recipe calls for 1 cup of flour to serve 8 people?

Solution

Students can use a ratio table to repeatedly add 1 cup of flour per 8 people to find the correct amount for 32 people.

Cups of Flour	1	2	3	4
Number of People	8	16	24	32

Problem
Students can also use the ratio table to see that multiplying by the ratio 4/4 (i.e., four times the recipe) provides the amount of flour needed for 32 people. Alternatively, the number of people served is always 8 times the number of cups of flour needed; thus, the ratio between them is 1:8.

Note. Adapted from Example 9 on page 39 and Example 10 on page 40 of the practice guide (Siegler et al., 2010).

3. Provide opportunities for students to use and discuss alternative strategies for solving ratio, rate, and proportion problems.

Instructional strategies from the examples:

- Focus instruction on the meaningful features of different problem types so students can transfer their learning to new situations.
- Help students identify key information needed to solve a problem and how to use diagrams or pictures to depict that information.
- Encourage students to use different diagrams and strategies to arrive at solutions.
- Provide opportunities for students to compare and discuss their diagrams and strategies.
- Provide real-life contexts in problems.

Teachers should focus instruction on meaningful features of different ratio, rate, and proportion problems to help students identify problems with common underlying structures. The goal is for students to transfer their learnings to new situations and contexts. For example, a recipe problem might call for 3 eggs to make 20 cupcakes and ask students to find the number of eggs for 80 cupcakes, while in another problem, building 3 doghouses requires 42 boards, and students need to determine how many boards are needed for 9 doghouses.

To develop students' ability to generalize to other contexts, teachers should first help them identify key information they will need to solve the problem. Then, teachers can teach students how to use diagrams to represent that information, focusing the diagrams on not only depicting the information but also the relationships between different quantities in the problem. Teachers should also provide opportunities for students to compare and discuss their various diagrams and strategies.

Finally, teachers should set ratio, rate, and proportion problems within real-life contexts, such as unit price, scaling, recipes, mixture, and time/speed/distance.

Potential Roadblocks

Many students misapply the cross-multiplication strategy.

Suggested Approach. Teachers can carefully present several examples of why cross-multiplication works, following the process in Example 2, to help students understand the logic behind the procedure. This will also help students see why the correct form of a ratio problem is necessary for the procedure to work.

Some students rely nearly exclusively on the cross-multiplication strategy for solving ratio, rate, and proportion problems, failing to recognize that there often are more efficient ways to solve these problems.

Suggested Approach. Teachers should encourage a variety of strategies for solving ratio, rate, and proportion problems. Presenting problems that are easier to solve with strategies other than cross-multiplication encourages students to use prior knowledge. For example, to find the solution for $5/15=6/x$, students can see that 15 is a multiple of 5 (i.e., $5\times3=15$), so x will be the same multiple of 6 (i.e., $x=6\times3$).

Requiring students to solve problems mentally can also help them see and use other strategies, as well as build number sense.

Students do not generalize strategies across different ratio, rate, and proportion contexts.

Suggested Approach. When presenting problems set in different contexts, teachers should also make sure to link these new problems with problems students have previously solved. Teachers can also encourage students to judge whether the same solution strategy can be used for different types of problems (e.g., recipe and mixture problems can be organized the same way so solutions can be compared side by side).

Reference

Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., Thompson, L., & Wray, J. (2010). *Developing effective fractions instruction for kindergarten through 8th grade* (NCEE 2010-4039). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/15