MARZANO RESEARCH

Math Strategies

Assisting Students Struggling with Mathematics

Grades K–8th

RECOMMENDATION 4

Interventions should include instruction on solving word problems that is based on common underlying structures.

RECOMMENDATION 5

Intervention materials should include opportunities for students to work with visual representations of mathematical ideas and interventionists should be proficient in the use of visual representations of mathematical ideas.

RECOMMENDATION 6

Interventions at all grade levels should devote about 10 minutes in each session to building fluent retrieval of basic arithmetic facts.

This document provides a summary of recommendations 4–6 from the WWC Practice Guide *Assisting Students Struggling with Mathematics: Response to Intervention (RtI) for Elementary and Middle Schools* (Gersten et al., 2009).

Content Area	Level of Evidence	Grades
Math	Strong	K, 1st, 2nd, 3rd, 4th, 5th, 6th, 7th, 8th

RECOMMENDATION 4

Interventions should include instruction on solving word problems that is based on common underlying structures.

Students who struggle with mathematics often face even larger difficulties in solving word problems. Thus, teachers should provide systematic, explicit instruction on solving word problems that is based in the problems' underlying structure. When students are taught the underlying structure of a word problem, they not only have greater success in problem-solving but can also gain insight into the deeper mathematical ideas in word problems. In this, teachers should make explicit connections between the structures of familiar and unfamiliar problems to help students learn when to apply a previously learned solution strategy.

Strategies

1. Teach students about the structure of various problem types, how to categorize problems based on structure, and how to determine appropriate solutions for each problem type.

Instructional strategies from the examples:

- Explicitly teach the important features of groups of problems with similar mathematical structures.
- Use visualizations to help students identify similarities in the structure of problems to determine appropriate solutions.

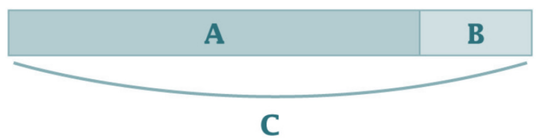
For groups of problems with similar mathematical structures, referred to as problem types, teachers should explicitly teach students the important underlying structural features that relate problems in the same problem type. Change problems and compare problems are two examples (see Example 1). Change problems always include a time element, and students use addition or subtraction to determine how much more or less. Compare problems focus on making comparisons between two different sets, and students need to calculate the unknown difference, unknown compared amount, or unknown referent amount.

To build understanding of each problem type, teachers should initially teach solution rules (i.e., guiding questions that lead to a solution equation) for each problem type through fully and partially worked examples, followed by student practice in pairs. Additionally, using visual representations can help students see the problem structure and determine an appropriate solution method.

Example 1. Sample Problem Types

Change Problems

The two problems here are addition and subtraction problems that students may be tempted to solve using an incorrect operation. In each case, students can draw a simple diagram like the one shown below, record the known quantities (two of three of A, B, and C), and then use the diagram to decide whether addition or subtraction is the correct operation to use to determine the unknown quantity.



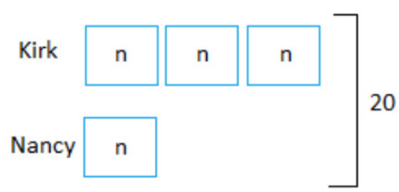
Problem 1. Brad has a bottlecap collection. After Madhavi gave Brad 28 more bottlecaps, Brad had 111 bottlecaps. How many bottlecaps did Brad have before Madhavi gave him more?

Problem 2. Brad has a bottlecap collection. After Brad gave 28 of his bottlecaps to Madhavi, he had 83 bottlecaps left. How many bottlecaps did Brad have before he gave Madhavi some?

Compare Problems

Problem. Kirk has 3 times as many baseball cards as Nancy.
Together, they have 20 baseball cards. How many cards does Kirk have?

Visual representation.



Note. Adapted from Example 1 on page 27 and Example 2 on page 28 of the practice guide (Gersten et al., 2009).

2. Teach students to recognize the common underlying structure between familiar and unfamiliar problems and to transfer known solution methods from familiar to unfamiliar problems.

Instructional strategies from the examples:

- Explicitly show students that not all pieces of information in the problem may be important for identifying the underlying structure.
- Provide opportunities for students to explain and discuss why a piece of information is relevant or irrelevant.

Superficial changes (e.g., format, key vocabulary, the inclusion of irrelevant information) can often lead students to see a familiar problem as one that is new and unfamiliar. Format changes might be something like presenting a problem as an advertisement in a brochure rather than in a traditional paragraph form. Changes in key vocabulary might be identifying a fraction as half, one half, or 1/2. These superficial changes are irrelevant to solving the problem but often lead to students struggling with identifying common underlying structures between new and old problems.

To facilitate transfer of methods to new problems from old problems, teachers should explicitly show students that not all pieces of information are relevant to identifying the underlying structure. Teachers should also provide opportunities for students to explain and discuss why a certain piece of information is relevant or irrelevant.

Example 2. Different Problems with the Same Strategy

Problem 1. Mike wants to buy 1 pencil for each of his friends. Each packet of pencils contains 12 pencils. How many packets does Mike have to buy to give 1 pencil to each of his 13 friends?

Problem 2. Mike wants to buy 1 pencil for each of his friends. Sally wants to buy 10 pencils. Each box of pencils contains 12 pencils. How many boxes does Mike have to buy to give 1 pencil to each of his 13 friends?

Discussion. The structure of these two problems is identical, as is the required solution. However, the inclusion of irrelevant information in the second problem (i.e., Sally wants to buy 10 pencils) may cause students to see these two problems as different. Teachers should provide an opportunity for students to discuss what information in the problem is relevant to finding a solution or, if the students struggle to identify what is relevant and irrelevant, explicitly show students the relevant information.

Note. Adapted from Example 3 on page 39 of the practice guide (Gersten et al., 2009).

Potential Roadblocks

The curricular material may not classify problems into problem types.

Suggested Approach. The key issue is determining the problem types and an instructional sequence for teaching them so that students understand a set of problem structures and the related mathematics. To accomplish this, teachers may need assistance from a mathematics coach, a mathematics specialist, or a district or state curriculum guide.

As problems get complex, so will the problem types and the task of discriminating among them.

Suggested Approach. As problems become more complex (e.g., multistep problems), teachers may need to explicitly and systematically teach students how to differentiate one problem type from another. Again, teachers themselves may need additional support in determining problem types, justifying their responses, and explaining and modeling problem types to students.

Content Area	Level of Evidence	Grades
Math	Moderate	K, 1st, 2nd, 3rd, 4th, 5th, 6th, 7th, 8th

RECOMMENDATION 5

Intervention materials should include opportunities for students to work with visual representations of mathematical ideas and interventionists should be proficient in the use of visual representations of mathematical ideas.

Many students struggle with mathematics as a result of not having a strong understanding of the relationships between the abstract symbols used in mathematics and various visual representations. Helping students develop their ability to express mathematical ideas using visual representations, then correctly convert this into symbols, is critical. As such, teachers should systematically teach students how to both develop visual representations and translate these into standard mathematical symbols they will use in solving the problem.

Strategies

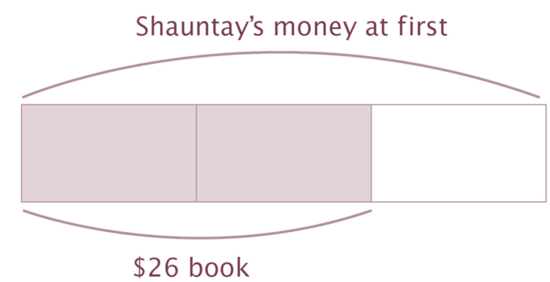
1. Use visual representations such as number lines, arrays, and strip diagrams.

Instructional strategies from the examples:

- Use visual representations (e.g., number lines, diagrams, pictorial representations) extensively and consistently.
- Explicitly link visual representations with standard mathematical symbols.

In early grades, teachers can use number lines, number paths, and other pictorial representations to help students develop foundational skills and procedural operations for counting, addition, and subtraction. In upper grades, diagrams and pictorial representations can be used to teach fractions or help students make sense of the underlying mathematical structure in word problems.

Example 1. Using Strip Diagrams to Make Sense of Fractions

Problem
Shauntay spent 2/3 of the money she had on a book that cost $26. How much money did Shauntay have before she bought the book?
Solution
Strip diagrams (also called model diagrams and bar diagrams) are one type of diagram that can be used to represent and help solve this problem. The full rectangle (consisting of all three equal parts joined together) represents Shauntay's money before she bought the book. Since she spent 2/3 of her money on the book, two of the three equal parts represent the $26 she spent on the book.

Two parts of the strip diagram represent the $26 spent on the book. That means that one part would equal $13 ($26 ÷ 2 = $13). To find the amount she had at first, we would multiply $13 by 3. So Shauntay had $39 before she bought the book.

Note. Adapted from Example 6 on page 34 of the practice guide (Gersten et al., 2009).

2. If visuals are not sufficient for developing accurate abstract thought and answers, use concrete manipulatives first. Although this can also be done with students in upper elementary and middle school grades, use of manipulatives with older students should be expeditious because the goal is to move toward understanding of—and facility with—visual representations, and finally, to the abstract.

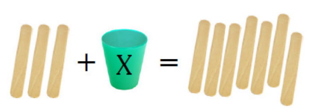
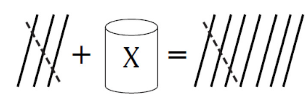
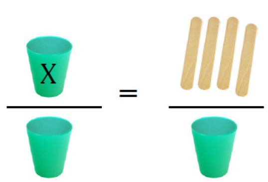
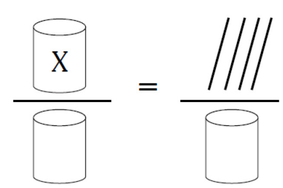
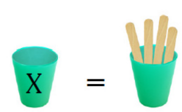
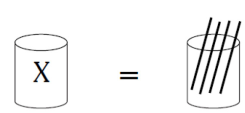
Instructional strategies from the examples:

- Use manipulatives, when necessary, and focus on systematically phasing them out.

Teachers should use concrete objects when using visual representations doesn't sufficiently support students understanding the more abstract symbols in math. However, teachers should focus on systematically fading the use of manipulatives to guide students toward reaching the more abstract level of using mathematical symbols.

This involves explicitly teaching the concepts and operations with the concrete manipulatives and consistently connecting the visual with the abstract levels. Teachers should use consistent language across the different representations (concrete manipulatives, visual representation, and abstract symbols) to solidify connections for students.

Example 2. A Set of Matched Concrete, Visual, and Abstract Representations to Teach Solving Single-Variable Equations

Solving the Equation with Concrete Manipulatives (Cups and Sticks)	Solving the Equation with Visual Representations of Cups and Sticks	Solving the Equation with Abstract Symbols
A 		$3 + 1x = 7$
B 		$-3 - 3$
C D 		$\frac{1x}{1} = \frac{4}{1}$
E 		$x = 4$

Note. Taken from Example 8 on page 35 of the practice guide (Gersten et al., 2009).

Potential Roadblocks

Many intervention materials provide very few examples of the use of visual representations.

Suggested Approach. Because many curricular materials do not include sufficient examples of visual representations, the interventionist may need the help of the mathematics coach or other teachers in developing the visuals. District staff can also arrange for the development of these materials for use throughout the district.

Some teachers or interventionists believe that instruction in concrete manipulatives requires too much time.

Suggested Approach. Expeditious use of manipulatives cannot be overemphasized. Since tiered interventions often rely on foundational concepts and procedures, the use of instruction at the concrete level allows for reinforcing and making explicit the foundational concepts and operations. Note that overemphasis on manipulatives can be counterproductive because students manipulating only concrete objects may not be learning to do math at an abstract level. The interventionist should use manipulatives in the initial stages strategically and then scaffold instruction to the abstract level. So, although it takes time to use manipulatives, this is not a major concern because concrete instruction will happen only rarely and expeditiously.

Some interventionists may not fully understand the mathematical ideas that underlie some of the representations. This is likely to be particularly true for topics involving negative numbers, proportional reasoning, and interpretations of fractions.

Suggested Approach. It is perfectly reasonable for districts to work with a local university faculty member, high school mathematics instructor, or mathematics specialist to provide relevant mathematics instruction to interventionists so they feel comfortable with the concepts. This can be coupled with professional development that addresses ways to explain these concepts in terms their students will understand.

Content Area	Level of Evidence	Grades
Math	Moderate	K, 1st, 2nd, 3rd, 4th, 5th, 6th, 7th, 8th

RECOMMENDATION 6

Interventions at all grade levels should devote about 10 minutes in each session to building fluent retrieval of basic arithmetic facts.

Students with difficulties in mathematics often demonstrate lack of fluency with quick retrieval of basic arithmetic fact (e.g., $3 \times 9 = \_\_$ and $11 - 7 = \_\_$). This weak ability is likely to inhibit students' understanding of deeper mathematics concepts (e.g., rational numbers, the commutative property).

For struggling students, devoting 10 minutes each day to building proficiency with quick retrieval of arithmetic facts can enhance their ability to grasp more advanced mathematics concepts.

Strategies

1. Provide about 10 minutes per session of instruction to build quick retrieval of basic arithmetic facts. Consider using technology, flash cards, and other materials for extensive practice to facilitate automatic retrieval.

Instructional strategies from the examples:

- Present facts in number families.
- Integrate previously learned facts into practice activities.
- Provide enough practice so retrieval becomes automatic.

Interventions at all grade levels should devote about 10 minutes in each session to building fluent retrieval of basic arithmetic facts.

The goal for students is the quick retrieval of facts without the use of pencil and paper or manipulatives. Teachers should present facts in number families (e.g., $7 \times 8 = 56$, $8 \times 7 = 56$, $56 \div 7 = 8$, and $56 \div 8 = 7$) to help build student's fluency. This also helps students learn about inverse operations. Additionally, teachers should incorporate cumulative review into these activities, integrating previously learned facts into students' practice activities and providing enough practice so retrieval for students become automatic.

Note that the panel that developed the practice guide acknowledges that students who are proficient in grade-level mathematics may not need to practice each session, but might still benefit from periodic, cumulative review.

2. Teach students in grades 2 through 8 how to use their knowledge of properties, such as commutative, associative, and distributive law, to derive facts in their heads.

(Note: This is really step 3 in the practice guide, as step 2 focuses on K–2).

Instructional strategies from the examples:

- Guide students to use properties of arithmetic (e.g., composition, decomposition, distributive property) to solve complex facts involving multiplication and division.

Rather than solely rely on rote memorization of facts, teachers should guide students to use what they know about properties of mathematics to master more complex facts about multiplication and division. Teachers can teach students how to use composition and decomposition, as well as the distributive property, to help increase students' facility with retrieving multiplication facts more quickly.

Example of Using Mathematical Properties to Support Multiplication Facts

To understand and quickly produce the seemingly difficult multiplication fact $13\times7=$, students recall that $13=10+3$, something they should have been taught consistently during their elementary career. Then, since $13\times7=(10+3)\times7=10\times7+3\times7$, the fact is parsed into easier, known problems $10\times7=$ and $3\times7=$ by applying the distributive property. Students can then rely on the two simpler multiplication facts (which they had already acquired) to quickly produce an answer mentally.

Note. Adapted from example in text on page 39 in the practice guide (Gersten et al., 2009).

Potential Roadblocks

Students may find fluency practice tedious and boring.

Suggested Approach. Games that provide students with the opportunity to practice new facts and review previously learned facts by encouraging them to beat their previous high score can help the practice become less tedious. Players may be motivated when their scores rise and the challenge increases.

Curricula may not include enough fact practice or may not have materials that lend themselves to teaching strategies.

Suggested Approach. Some contemporary curricula deemphasize fact practice, so this is a real concern. In this case, teachers should consider using a supplemental program, based in either flash cards or technology.

Reference

Gersten, R., Beckmann, S., Clarke, B., Foegen, A., Marsh, L., Star, J. R., & Witzel, B. (2009). *Assisting students struggling with mathematics: Response to Intervention (RtI) for elementary and middle schools* (NCEE 2009-4060). U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance. https://ies.ed.gov/ncee/wwc/PracticeGuide/2